

Analytic Proof Theory for Rational Closure

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Abstract. Non-monotonic reasoning allows intelligent agents to draw plausible conclusions from incomplete information while revising them in light of exceptional circumstances or newly acquired knowledge. In symbolic approaches to this problem, one distinguishes between local non-monotonicity, where more specific information may override general rules, and global non-monotonicity, where extending a knowledge base may invalidate previous conclusions. Rational closure is a prominent formalism combining these two forms of reasoning and a standard reference point for conditional entailment. While this formalism is well understood semantically and computationally, its proof theory remains less developed. We introduce the first analytic proof calculus for rational closure, providing a syntactic account of how the conclusions are inferred. The calculus is modular: a base layer captures globally monotonic KLM-style entailment, while a single additional rule expresses the model-minimization principle responsible for global non-monotonicity. It allows optimal-complexity proof search, which yields either a derivation of the queried entailment or a saturated branch witnessing its failure. The completeness proof further connects these proof-theoretic witnesses with their constructive and semantic counterparts, closely relating the three complementary perspectives on rational closure.

Keywords: Proof calculi · Rational closure · Non-monotonic reasoning.

1 Introduction

Intelligent agents are often required to draw conclusions from incomplete information while remaining able to revise these conclusions in the presence of new evidence or exceptional circumstances. The classical symbolic approach to this problem represents knowledge by means of *conditionals* of the form $A \rightsquigarrow B$, understood as “if A holds, then normally B holds”, while allowing exceptions. Conditionals naturally arise in commonsense reasoning, reasoning about actions, normative reasoning, and other areas of knowledge representation concerned with defeasible inference.

The most influential semantic approach to conditional reasoning is based on the idea of preference: rather than considering all situations satisfying the antecedent equally, it evaluates conditionals with respect to the most typical (or otherwise preferred) ones. This intuition is formalized by *preferential semantics* [30]. A major milestone in its development is the seminal work of Kraus, Lehmann, and Magidor [24], which identified fundamental principles of non-monotonic reasoning (now known as KLM principles) and established their correspondence with preferential models. Their framework has

since become one of the cornerstones of conditional reasoning, with preferential entailment serving as one of its central reference notions. Preference-based semantics also underlies several important logical traditions arising from different interpretations of preference, including Åqvist’s logics of conditional obligation [1] and Lewis’ logics of counterfactuals [27], where preference is interpreted in terms of moral desirability and comparative similarity, respectively.

The above preference-based formalisms all share the same form of defeasibility, namely *local non-monotonicity* [2], where more specific information may override general conditional rules. A conceptually different phenomenon is *global non-monotonicity*, where newly acquired knowledge may invalidate conclusions previously derived from the knowledge base. To incorporate this second form of defeasibility within preferential semantics, Lehmann and Magidor [26] proposed *rational closure*, a refinement of preferential entailment defined by minimal models in a certain subclass of preferential models, called *ranked models*. Rational closure has since become one of the standard reference formalisms for conditional reasoning, giving rise to numerous refinements and extensions (e.g. [25, 8, 12, 9]) and serving as the basis of several approaches to defeasible reasoning in description logics (e.g. [13, 3, 10]).

The broad applicability and rich landscape of preference-based conditional formalisms have also motivated extensive research in their proof theory, syntactically characterizing these formalisms in terms of inference rules and derivations. Such characterisations make the structure of reasoning explicit, facilitating systematic comparison between different logics and modular extensions of existing formalisms, automated reasoning, and the use of derivations as formal certificates and explanations. Particularly appealing are *analytic* proof calculi, such as sequent and tableau systems, whose derivations are restricted to formulas already occurring in the input, making them well suited for structural analysis and systematic proof search. Consequently, a broad range of analytic proof calculi has been developed for globally monotonic preference-based conditional logics, including several calculi for preferential entailment [20, 11, 14] and related normative [16, 15], and counterfactual logics [23, 22].

By contrast, the proof theory of preference-based systems exhibiting global non-monotonicity, such as rational closure, remains largely underexplored, despite decades of research studying them from semantic and constructive perspectives. While analytic proof calculi have been developed for non-preferential formalisms exhibiting global non-monotonicity, such as default logic [4, 29], circumscription [5], and autoepistemic logic [28], capturing the minimization of preferential models underlying rational closure is considerably more challenging. To the best of our knowledge, the only proof-theoretic characterization of rational closure is due to Straßer [32], formulated within the framework of adaptive logic, where global non-monotonicity is represented by *dynamic* derivations, allowing conclusions to be retracted in light of later reasoning steps. While highly expressive, such dynamic derivations differ fundamentally from the static, local nature of analytic proof calculi and are therefore less suitable for proof search, structural analysis, and explanatory purposes.

In this paper, we introduce the first analytic proof-theoretic characterization of rational closure in the form of a tableau calculus. The calculus is developed in two stages by adapting established proof-theoretic techniques to reasoning over ranked models.

First, we enrich an existing analytic calculus for preferential entailment with analytic cuts [31], allowing derivations to make explicit the relative ranks of antecedents in the knowledge base. We then use this information to internalize the model-minimality requirement of rational closure in a single additional inference rule. The resulting calculus provides a modular account of both preferential entailment and rational closure, supporting globally monotonic and globally non-monotonic reasoning within a common framework, with the transition governed by one rule.

Analyticity supports terminating, easily automatable proof search, producing either a derivation of polynomially bounded branch depth or a syntactic witness of non-derivability in the form of a saturated tableau branch. The completeness proof further establishes a three-way correspondence between these proof-theoretic witnesses and their constructive and semantic counterparts. Since each rule represents a simple step of reasoning over ranked models, derivations can also explain entailment by exposing how a conclusion is obtained. This complements justification-based explanations for rational closure [18], which instead identify the parts of the knowledge base responsible for the entailment. Finally, the modular calculus provides a framework for developing automated deduction strategies combining globally monotonic methods for preferential entailment with non-monotonic reasoning for rational closure.

The remainder of the paper is structured as follows. Section 2 introduces conditional bases and ranked-model semantics and uses a running example to motivate the distinction between preferential entailment and rational closure. Section 3 develops the globally monotonic layer, which captures preferential entailment while deriving the relative-rank information needed later. Section 4 adds a single rule using this information to enforce model minimality, thereby obtaining a proof system for rational closure, and proves completeness by connecting its proof-theoretic, constructive, and semantic characterizations. Section 5 analyses proof search, showing that suitably restricted rule applications yield polynomially bounded branches and complexity-optimal decision procedures for derivability. Section 6 concludes and outlines directions for future work.

2 Preliminaries

Let $\mathbb{P}$ be a finite set of propositional variables, and let $\mathcal{L}^{\mathbb{P}}$ be the language of propositional formulas over $\mathbb{P}$, which we will denote by capital Latin letters. A *finite conditional knowledge base* (or simply a *base*) is a finite set of *conditionals* of the form $A \rightsquigarrow B$, where $A, B \in \mathcal{L}^{\mathbb{P}}$. The central problem of conditional reasoning is to determine which conditionals are entailed by a given base. We focus on two foundational entailment operators, *preferential entailment* [24] and *rational closure* [26]. This section introduces them through their standard semantic and constructive characterizations.

Ranked models Many conditional formalisms are based on the idea of preference: rather than treating all situations satisfying the antecedent equally, they evaluate a conditional with respect to the most typical, desirable, or otherwise preferred ones. This intuition is formalized using *preferential possible-world models*, in which possible worlds are ordered by a preference relation. A particularly important subclass is given by *ranked models*, where the preference relation is total: they equivalently characterize

preferential entailment and form the semantic basis of rational closure. We use the following definition of ranked models from [26] with two adjustments: modular orders are equivalently represented as strict versions of total preorders (as usual in conditional logics), and smoothness is replaced by well-foundedness, which does not affect the entailment for finite bases [26].

Definition 1 (Ranked models). A ranked model is a tuple $\langle W, \preceq, V \rangle$, where W is a set of possible worlds, $\preceq$ is a reflexive, transitive, and total relation on W , and V is a valuation function mapping each world to the set of propositional atoms it satisfies. $\prec$ denotes the strict version of $\preceq$, defined by $w_1 \prec w_2$ iff $w_1 \preceq w_2$ and $w_2 \not\preceq w_1$. Ranked models additionally require well-foundedness, i.e. the absence of infinite $\prec$ -descending chains. A formula A is satisfied in a world w of model M (denoted $M, w \models A$) when $V(w) \models A$. A conditional $A \rightsquigarrow B$ is satisfied in a ranked model M (denoted $M \models A \rightsquigarrow B$) when $M, u \models B$ for every $u \in W$ such that $M, u \models A$ and there is no $w \prec u$ with $M, w \models A$. A ranked model M is called a model of base $\mathcal{K}$ (denoted $M \models \mathcal{K}$) when it satisfies all conditionals in it.

The preference relation induces the rank function on worlds, defined recursively by $\mathcal{R}^M(w) = \sup\{\mathcal{R}^M(u) + 1 \mid u \prec w\}$, with $\sup \emptyset = 0$. As is usual in work on rational closure for finite bases, we restrict attention to naturally ranked models, i.e. models whose worlds have finite ranks in $\mathbb{N}$. The rank of a valuation v is then defined as $\mathcal{R}^M(v) = \min(\{\infty\} \cup \{\mathcal{R}^M(w) \mid V(w) = v\})$, and the rank of a formula as $\mathcal{R}^M(A) = \min(\{\infty\} \cup \{\mathcal{R}^M(w) \mid M, w \models A\})$. Note that with this definition $M \models A \rightsquigarrow B$ iff $\mathcal{R}^M(A) < \mathcal{R}^M(A \wedge \neg B)$ or $\mathcal{R}^M(A) = \infty$.

Preferential entailment The most widely studied notion of entailment based on preferential semantics is *preferential entailment* operator, introduced by Kraus, Lehmann, and Magidor [24]. Although originally defined over the more general class of preferential models, it was later shown to coincide with entailment over the class of ranked models [26].

Definition 2 (Preferential entailment). A base $\mathcal{K}$ preferentially entails a conditional $X \rightsquigarrow Y$ when every ranked model satisfying all conditionals in $\mathcal{K}$ also satisfies $X \rightsquigarrow Y$.

Example 1. We will use the classical “penguin triangle” from [26] as the running example. Consider the base $\mathcal{K}_\Delta = \{\mathbf{b} \rightsquigarrow \mathbf{f}, \mathbf{p} \rightsquigarrow \neg \mathbf{f}, \mathbf{p} \rightsquigarrow \mathbf{b}\}$, encoding the statements “birds normally fly”, “penguins normally do not fly”, and “penguins normally are birds”. Preferential entailment derives the conditional $\mathbf{b} \rightsquigarrow \neg \mathbf{p}$ (“normal birds are not penguins”). Indeed, if a minimal $\mathbf{b}$ -world satisfied $\mathbf{p}$, then, by the first two conditionals, it could not be a minimal $\mathbf{p}$ -world. Hence there would exist a minimal $\mathbf{p}$ -world below it, which, by the third conditional, would also satisfy $\mathbf{b}$, contradicting the assumption that the original world was a minimal $\mathbf{b}$ -world.

Note that preferential entailment is *globally monotonic*, i.e. its conclusions cannot be invalidated by extending the knowledge base. Consequently, it must reason overly cautiously about more specific antecedents. E.g., in the above example, $\mathcal{K}_\Delta$ does not preferentially entail $(\mathbf{b} \wedge \mathbf{g}) \rightsquigarrow \mathbf{f}$ (“green birds normally fly”), even though the base contains no information relating greenness to the ability to fly. This limitation motivates stronger, globally non-monotonic entailment operators.

Rational closure To handle limitations of preferential entailment, Lehmann and Magidor [26] introduced rational closure as the minimal extension of a base satisfying KLM postulates and the following rational monotonicity principle: if it contains $(A \rightsquigarrow C)$ and does not contain $(A \rightsquigarrow \neg B)$, then it should contain $(A \wedge B \rightsquigarrow C)$. They showed that such an extension exists for a large class of conditional bases (including all finite ones) and provided an explicit construction based on the following notion of *exceptionality ranks*.

Definition 3 (Exceptionality ranks). *Formula A is called exceptional for a base $\mathcal{K}$ when $\overline{\mathcal{K}}$ classically entails $\neg A$, where $\overline{\mathcal{K}}$ is the materialization of $\mathcal{K}$ defined as*

$$\overline{\mathcal{K}} = \{A \rightarrow B \mid (A \rightsquigarrow B) \in \mathcal{K}\}.$$

The stratification $RSeq(\mathcal{K})$ of $\mathcal{K}$ w.r.t. exceptionality ranks is a sequence $\{C_i\}_{i=0}^\infty$ where $C_0 = \mathcal{K}$ and $C_{i+1} = \{(A \rightsquigarrow B) \in C_i \mid A \text{ is exceptional for } C_i\}$. The exceptionality rank of a formula A is defined as

$$\mathcal{R}_{\mathcal{K}}^{ex}(A) = \min\{i \mid A \text{ is not exceptional for } C_i\},$$

with $\mathcal{R}_{\mathcal{K}}^{ex}(A) = \infty$ if A is exceptional for every C_i .

The rational closure can then be defined as the set of conditionals $A \rightsquigarrow B$ for which A is less exceptional than $A \wedge \neg B$ or has infinite exceptionality rank.

Definition 4 (Rational closure [26]). *Conditional $X \rightsquigarrow Y$ belongs to the rational closure of a base $\mathcal{K}$ (denoted $RC(\mathcal{K})$) when*

$$\mathcal{R}_{\mathcal{K}}^{ex}(X) < \mathcal{R}_{\mathcal{K}}^{ex}(X \wedge \neg Y) \text{ or } \mathcal{R}_{\mathcal{K}}^{ex}(X) = \infty.$$

Example 2. The base $\mathcal{K}_\Delta$ from Example 1 has the following stratification:

$$C_0 = \mathcal{K}, C_1 = \{\mathbf{p} \rightsquigarrow \neg \mathbf{f}, \mathbf{p} \rightsquigarrow \mathbf{b}\}, \text{ and } C_k = \emptyset \text{ for } k > 1.$$

Accordingly, $\mathcal{R}_{\mathcal{K}_\Delta}^{ex}(\mathbf{b}) = \mathcal{R}_{\mathcal{K}_\Delta}^{ex}(\mathbf{b} \wedge \mathbf{g}) = 0$, $\mathcal{R}_{\mathcal{K}_\Delta}^{ex}(\mathbf{p}) = \mathcal{R}_{\mathcal{K}_\Delta}^{ex}(\mathbf{p} \wedge \neg \mathbf{f}) = \mathcal{R}_{\mathcal{K}_\Delta}^{ex}(\mathbf{b} \wedge \mathbf{g} \wedge \neg \mathbf{f}) = 1$, so $((\mathbf{b} \wedge \mathbf{g}) \rightsquigarrow \mathbf{f}) \in RC(\mathcal{K}_\Delta)$, but $(\mathbf{p} \rightsquigarrow \mathbf{f}) \notin RC(\mathcal{K}_\Delta)$.

Definition 4 also provides an efficient decision procedure for rational closure: since in Definition 3 $C_{i+1} \subseteq C_i$, the sequence $RSeq(\mathcal{K})$ reaches a fixed point (which we will denote by $C_\infty = \bigcap_{i \in \mathbb{N}} C_i$) for any finite $\mathcal{K}$ in at most $|\mathcal{K}|$ steps. Therefore, the prefix of $RSeq(\mathcal{K})$ up to the fixed point can be computed directly by definition via a polynomial number of classical entailment checks. The computation of $\mathcal{R}_{\mathcal{K}}^{ex}(A)$ can then also be done by a polynomial number of classical entailment checks, resulting in a procedure deciding membership in $RC(\mathcal{K})$ with complexity P^{NP} . It was later shown that membership in rational closure is actually P^{NP} -complete [17].

Subsequently, Giordano et al. [21] gave an equivalent semantic characterization of rational closure in terms of minimal ranked models. Their original characterization minimizes world ranks pointwise among models of the base with the same worlds and valuations (so called Fixed Interpretations Minimal Semantics) together with a restriction to canonical models containing worlds for all valuations compatible with the

base. In later work (e.g. [12, 7, 9]), the same minimal-model semantics is commonly presented by directly assigning a rank in $\mathbb{N} \cup \{\infty\}$ to every valuation and minimizing these ranks pointwise. In this presentation, no separate canonicity condition is needed, since all valuations are already included in the domain of the ranking. We use this later valuation-based definition of minimality, although we retain possible worlds and allow several worlds to have the same valuation. This is only a representational difference: model comparison and satisfaction of conditionals depend only on the lowest rank assigned to each valuation, rather than on individual worlds.

Definition 5 (Minimal ranked models). *A ranked model M of $\mathcal{K}$ is minimal for $\mathcal{K}$ if there is no other ranked model M' of $\mathcal{K}$ such that $M' \prec_{Mod} M$, where $M' \preceq_{Mod} M$ iff, for every valuation v , $\mathcal{R}^M(v) = \infty$ or $\mathcal{R}^{M'}(v) \leq \mathcal{R}^M(v)$, and $M' \prec_{Mod} M$ iff $M' \preceq_{Mod} M$ and $M \not\preceq_{Mod} M'$.*

For finite bases, minimal ranked models under this valuation-based ordering exist and characterize rational closure; moreover, in these models formula ranks coincide with exceptionality ranks [21, 12].

Theorem 1. *For any ranked model M minimal for $\mathcal{K}$, $\mathcal{R}^M(A) = \mathcal{R}_{\mathcal{K}}^{ex}(A)$ for any $A \in \mathcal{L}^{\mathcal{P}}$, and $(X \rightsquigarrow Y) \in RC(\mathcal{K})$ iff $X \rightsquigarrow Y$ is satisfied in M .*

3 Globally Monotonic Layer

This section develops the globally monotonic layer of our proof system. Its purpose is not to provide yet another calculus for preferential entailment, for which numerous proof systems already exist, but to construct a base calculus strong enough to support the globally non-monotonic layer introduced in the next section. Existing calculi are typically based on general preferential models, as in the seminal first tableau system of Giordano et al. [20], although proof systems have also been developed directly for ranked semantics, including calculi for ranked entailment **R** (not to be confused with rational closure) [20], Åqvist's logic **G** [16], and Lewis' logic **V** [23]. None of these systems, however, is designed to derive comprehensive information about the relative ranks of the antecedents occurring in the knowledge base, which will be essential for expressing model minimality.

We therefore introduce a new tableau-style calculus for ranked models, and hence for preferential entailment, extending standard preferential reasoning with *analytic cuts*. Their role is to force derivations to derive complete information about the relative ranks of antecedents in the knowledge base—precisely the information required by the globally non-monotonic layer introduced in the next section—while preserving analyticity and complexity-optimal proof search. Analytic cuts also allow us to avoid additional proof-theoretic machinery employed by other calculi for ranked models, such as labels [20], blocks [23], or non-standard truth conditions for conditionals [16], resulting in comparatively simple and readable derivations.

Following the canonical proof-theoretic approach [20] to preferential entailment, our calculus reasons over the modal language obtained by extending propositional formulas with the modality $\Diamond$, where $\Diamond A$ expresses that there exists a strictly preferred world

satisfying A . Intuitively, $\Diamond A$ states that the current world is exceptional for A . As observed in [20], a conditional $A \rightsquigarrow B$ is satisfied iff every world satisfies the formula $(A \wedge \neg \Diamond A) \rightarrow B$. We therefore reason directly in this modal language. Since nested occurrences and Boolean combinations of $\Diamond$ are unnecessary for conditional entailment, we restrict ourselves to the language $\mathcal{L}^\Diamond$ consisting of either propositional formulas or formulas of the form $\Diamond A$ or $\neg \Diamond A$ (we will denote them by small Greek letters).

Each derivation is carried out relative to a fixed background base $\mathcal{K}$ and performs a systematic search for a model of $\mathcal{K}$ satisfying the statement to be derived. It operates with *tableaux* — finite sets of formulas from $\mathcal{L}^\Diamond$, representing a world in a model satisfying all formulas in the tableau. While the calculus is formulated for arbitrary tableaux, satisfaction of a conditional $A \rightsquigarrow B$ in any model of $\mathcal{K}$ (and hence conditional entailment from $\mathcal{K}$) is expressed by derivability, i.e. non-satisfiability, of the tableau $\{A, \neg \Diamond A, \neg B\}$. Tableaux *derivations* are trees whose internal nodes are tableaux and whose leaves are either tableaux or the *concluding nodes* of one of three forms: $\boxtimes$, $Sat(\Gamma^P)$, and $UnSat(\Gamma^P)$, where Γ^P is a set of propositional formulas. Intuitively, $\boxtimes$ closes an unsuccessful search branch, while $Sat(\Gamma^P)$ and $UnSat(\Gamma^P)$ delegate the remaining purely propositional reasoning to classical satisfiability and unsatisfiability checking.

Unlike standard tableau systems, some of our rules inspect the entire tableau branch rather than only its leaf. This is necessary to capture the totality of the preference relation, similarly to the use of hypersequents in sequent calculi for total preference models [16]. Accordingly, our inference rules have the form

$$\frac{\Gamma_1 \& \dots \& \Gamma_n}{\Sigma_1 \mid \dots \mid \Sigma_m},$$

where each Γ_i is a tableau and each Σ_j is either a tableau or a concluding node. Such a rule states that whenever all tableaux $\Gamma_1, \dots, \Gamma_n$ occur on a branch, at least one of the conclusions must hold. Accordingly, an application of the rule extends every branch containing $\Gamma_1, \dots, \Gamma_n$ by adding the children $\Sigma_1, \dots, \Sigma_m$ to its leaf.

A derivation is *closed* when every leaf is either $\boxtimes$, $Sat(\Gamma^P)$ with Γ^P classically satisfiable, or $UnSat(\Gamma^P)$ with Γ^P classically unsatisfiable. We stop derivations at these concluding nodes for conciseness; they can equivalently be unfolded into derivations in standard proof [19] and refutation [6] calculi for classical logic.

The globally monotonic layer $\mathcal{T}_{\mathcal{A}}^R$ consists of seven rules, grouped according to their purpose (Fig. 1). *Concluding rules* close branches containing an inconsistent tableau: $(\Rightarrow)$ detects classical inconsistency among propositional formulas, while $(id\Diamond)$ detects contradictions of the form $\Diamond A \wedge \neg \Diamond A$. *Rank rules* reason about the relative ranks of worlds. Rule $(\mathcal{R}^\downarrow)$ states that every world satisfying $\Diamond A$ has a strictly preferred minimal A -world. Rule $(\mathcal{R}^\uparrow)$ states that for any worlds w_1 and w_2 satisfying $\Diamond A$ and $\neg \Diamond A$, respectively (and hence $w_2 \prec w_1$), if $M, w_1 \models \neg \Diamond B$, then $M, w_2 \models \neg B$. Finally, rule $(\leq)$ captures totality of the preference relation: for every two worlds, all $\Diamond$ -formulas falsified in one of them (the one with greater or equal rank) are also falsified in the other one. *Base rule* $(\rightsquigarrow)$ internalizes satisfaction of the background base $\mathcal{K}$: for every conditional $X \rightsquigarrow Y \in \mathcal{K}$, every world satisfying both X and $\neg \Diamond X$ (and thus minimal for X) also satisfies Y . Final rule (*acut*) is an analytic cut. It branches on whether a formula

Concluding rules

$$\frac{\Gamma}{UnSat(\Gamma \cap \mathcal{L}^{\mathbb{P}})} (\Rightarrow) \quad \frac{\Diamond A, \neg \Diamond A, \Gamma}{\boxtimes} (ID\Diamond)$$

Rank rules

$$\frac{\Diamond A, \Gamma}{A, \neg \Diamond A} (\mathcal{R}^\downarrow) \quad \frac{\Diamond A, \neg \Diamond B, \Gamma \ \& \ \neg \Diamond A, \Gamma'}{\neg B, \neg \Diamond A, \Gamma'} (\mathcal{R}^\uparrow) \quad \frac{\Gamma_1 \ \& \ \Gamma_2}{\Gamma_1^{-\Diamond}, \Gamma_2 \mid \Gamma_2^{-\Diamond}, \Gamma_1} (\lessapprox)$$

Base rule(for X, Y such that $X \rightsquigarrow Y \in \mathcal{K}$)

$$\frac{X, \neg \Diamond X, \Gamma}{Y, X, \neg \Diamond X, \Gamma} (\rightsquigarrow)$$

Analytic cut(for $\alpha \in \mathcal{F}(\Gamma_0)$)

$$\frac{\Gamma}{\alpha, \Gamma \mid \neg \alpha, \Gamma} (acut)$$

Fig. 1. Rules of the globally monotonic calculus $\mathcal{T}_{\mathcal{A}}^R$. Here, $\Gamma^{-\Diamond} = \{\neg \Diamond A \mid \neg \Diamond A \in \Gamma\}$.

α holds at a world, but only for a fixed set of formulas that occur in the initial tableau Γ_0 (possibly preceded by $\neg \Diamond$ or $\Diamond$). This set is defined as follows:

$$\mathcal{F}(\Gamma_0) = \{Z, \Diamond Z, \Diamond \neg Z \mid \{Z, \Diamond Z, \neg \Diamond Z\} \cap \Gamma_0 \neq \emptyset \text{ or } Z \in \bigcup_{X \rightsquigarrow Y \in \mathcal{K}} \{X, Y\}\}.$$

Since each rule directly captures a valid semantic property of ranked models, soundness of $\mathcal{T}_{\mathcal{A}}^R$ w.r.t. ranked models of $\mathcal{K}$ is immediate.

Theorem 2 (Soundness of $\mathcal{T}_{\mathcal{A}}^R$). *Every tableau derivable in $\mathcal{T}_{\mathcal{A}}^R$ for background base $\mathcal{K}$ is not satisfied by any ranked model of $\mathcal{K}$.*

Proof. By contraposition. Let Γ be satisfied in a ranked model M of $\mathcal{K}$. By the semantic validity of the non-concluding rules, from every rule application whose numerator tableaux are satisfied in M , at least one conclusion can be chosen so that satisfiability in M is preserved. Thus every derivation of Γ contains a branch consisting entirely of tableaux satisfied in M . This branch cannot terminate in $\boxtimes$, since no satisfiable tableau contains both $\Diamond A$ and $\neg \Diamond A$, nor in a correct *UnSat* node, since the propositional part of a satisfiable tableau is classically satisfiable. Hence the derivation cannot be closed.

The presented calculus is analytic in the sense that every derivation with root tableau Γ_0 relative to a base $\mathcal{K}$ contains, provided that double negations introduced by rule applications are immediately eliminated, only formulas from $\mathcal{F}^*(\Gamma_0) = \bigcup_{\alpha \in \mathcal{F}(\Gamma_0)} \{\alpha, \neg \alpha\}$.

Since $\mathcal{F}^*(\Gamma_0)$ is finite whenever Γ_0 and $\mathcal{K}$ are finite, every derivation without repeated tableaux is finite. Consequently, exhaustive application of non-concluding rules to any initial tableau Γ either produces a closed derivation or yields a *saturated* branch, which cannot be closed by any concluding rule and on which every non-concluding rule application has at least one conclusion already present. The latter case will be the key to completeness: from every saturated branch we reconstruct a ranked model satisfying $\mathcal{K}$ and the root tableau, simultaneously establishing completeness and providing a procedure for extracting countermodels from failed derivations.

Theorem 3 (Completeness of $\mathcal{T}_{\mathcal{A}}^R$). *Every tableau Γ_0 that is not satisfied in any ranked model of $\mathcal{K}$ has a closed $\mathcal{T}_{\mathcal{A}}^R$ -derivation relative to $\mathcal{K}$.*

Proof. Starting from Γ_0 , exhaustively apply the non-concluding rules without repeating tableaux. By finiteness, this yields either a closed derivation or a saturated branch. We show that, in the latter case, Γ_0 is satisfiable in a ranked model of $\mathcal{K}$. Let W consist of the subset-maximal tableaux occurring on the saturated branch. For each $\Gamma \in W$, choose a valuation $V(\Gamma)$ satisfying all formulas in $\Gamma \cap \mathcal{L}^{\mathbb{P}}$; such a valuation exists since the branch cannot be closed by $(\Rightarrow)$. Define $\Gamma_1 \preceq \Gamma_2$ iff $\Gamma_1^{\neg\Diamond} \supseteq \Gamma_2^{\neg\Diamond}$. The resulting model $M = \langle W, \preceq, V \rangle$ is reflexive and transitive, total by saturation under $(\preceq)$, and well-founded since W is finite. We prove the following truth lemma: for every $\Gamma \in W$ and every $\alpha \in \Gamma$, we have $M, \Gamma \models \alpha$. The propositional case follows from the choice of V . If $\Diamond A \in \Gamma$, saturation under $(\mathcal{R}^{\downarrow})$ yields a tableau containing A and $\neg\Diamond A$; extending it to a subset-maximal tableau $\Gamma' \in W$, saturation under $(\preceq)$ and the inapplicability of $(\text{id}\Diamond)$ give $\Gamma' \prec \Gamma$. Hence $M, \Gamma' \models A$. Suppose now that $\neg\Diamond A \in \Gamma$ and $\Gamma' \prec \Gamma$. Then $\Gamma'^{\neg\Diamond} \supseteq \Gamma^{\neg\Diamond}$, so some $\neg\Diamond B$ belongs to Γ' but not to Γ . Saturation under (acut) and subset-maximality therefore give $\Diamond B \in \Gamma$. Applying $(\mathcal{R}^{\uparrow})$ to $\Diamond B$, $\neg\Diamond A \in \Gamma$ and $\neg\Diamond B \in \Gamma'$ yields $\neg A \in \Gamma'$. Thus $M, \Gamma' \not\models A$, and consequently $M, \Gamma \models \neg\Diamond A$. It remains to show that M is a model of $\mathcal{K}$. Let $(X \rightsquigarrow Y) \in \mathcal{K}$ and let $\Gamma \in W$ be minimal among the worlds satisfying X . Saturation under (acut) , subset-maximality, and the truth lemma imply that $X, \neg\Diamond X \in \Gamma$. Saturation under $(\rightsquigarrow)$ then gives $Y \in \Gamma$, and hence $M, \Gamma \models Y$. Finally, Γ_0 is contained in some subset-maximal tableau on the branch, so the truth lemma shows that it is satisfied in M .

Example 3. An attempt to derive preferential entailment of $\mathbf{p} \rightsquigarrow \mathbf{f}$ in $\mathcal{T}_{\mathcal{A}}^R$ w.r.t. $\mathcal{K}_{\Delta}$ from Example 1 results in the following derivation, the leftmost branch of which is saturated (modulo some possible additional tableaux, which are subsumed by tableaux at the branch):

$$\begin{array}{c}
 \frac{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}}{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}} (\rightsquigarrow) \\
 \frac{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}}{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \Diamond\mathbf{b}} (\mathcal{R}^{\downarrow}) \qquad \frac{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \neg\Diamond\mathbf{b}}{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}} (\rightsquigarrow) \\
 \frac{\mathbf{b}, \neg\Diamond\mathbf{b}}{\mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}} (\rightsquigarrow) \qquad \frac{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}}{\text{UnSat}(\neg\mathbf{f}, \mathbf{f}, \dots)} (\Rightarrow) \\
 \frac{\mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}}{\mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}, \neg\mathbf{p}} (\mathcal{R}^{\uparrow}) \\
 \frac{\mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}, \neg\mathbf{p}}{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \Diamond\mathbf{b}, \neg\Diamond\mathbf{b}} (\preceq) \\
 \frac{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \Diamond\mathbf{b}, \neg\Diamond\mathbf{b}}{\Box} (\text{id}\Diamond)
 \end{array}$$

The saturated branch contains two subset-maximal tableaux: $\Gamma_1 = \{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \Diamond\mathbf{b}\}$ and $\Gamma_2 = \{\mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}, \neg\mathbf{p}, \neg\Diamond\mathbf{p}\}$. Using the construction from the proof above we obtain

the ranked model $M = \langle W, \preceq, V \rangle$ with $W = \{\Gamma_1, \Gamma_2\}$, $\Gamma_2 \prec \Gamma_1$, $V(\Gamma_1) = \mathbf{bp}$, $V(\Gamma_2) = \mathbf{bf}$.¹ It is easy to check that M is a model of $\mathcal{K}_\Delta$ but does not satisfy $\mathbf{p} \rightsquigarrow \mathbf{f}$.

Due to soundness and completeness w.r.t. all models of $\mathcal{K}$, $\mathcal{T}_{\mathcal{A}}^R$ captures preferential entailment by Definition 2.

Corollary 1. $\mathcal{K}$ preferentially entails $(X \rightsquigarrow Y)$ iff the tableau $\{X, \neg \Diamond X, \neg Y\}$ is derivable in $\mathcal{T}_{\mathcal{A}}^R$ relative to $\mathcal{K}$.

Example 4. The semantic argument from Example 1 can be encoded by the following closed derivation of $\{\mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}\}$ relative to $\mathcal{K}_\Delta$:

$$\begin{array}{c}
 \frac{\mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}}{\neg \Diamond \mathbf{p}, \mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}} \text{ (acut)} \\
 \frac{\neg \Diamond \mathbf{p}, \mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}}{\mathbf{f}, \neg \Diamond \mathbf{p}, \mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}} (\rightsquigarrow) \quad \frac{\Diamond \mathbf{p}, \mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}}{\mathbf{p}, \neg \Diamond \mathbf{p}} (\mathcal{R}^\downarrow) \\
 \frac{\mathbf{f}, \neg \Diamond \mathbf{p}, \mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}}{\neg \mathbf{f}, \mathbf{f}, \neg \Diamond \mathbf{p}, \mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}} (\rightsquigarrow) \quad \frac{\mathbf{p}, \neg \Diamond \mathbf{p}}{\mathbf{b}, \mathbf{p}, \neg \Diamond \mathbf{p}} (\rightsquigarrow) \\
 \frac{\neg \mathbf{f}, \mathbf{f}, \neg \Diamond \mathbf{p}, \mathbf{b}, \neg \Diamond \mathbf{b}, \mathbf{p}}{UnSat(\neg \mathbf{f}, \mathbf{f}, \dots)} (\Rightarrow) \quad \frac{\mathbf{b}, \mathbf{p}, \neg \Diamond \mathbf{p}}{\neg \mathbf{b}, \mathbf{b}, \mathbf{p}, \neg \Diamond \mathbf{p}} (\mathcal{R}^\uparrow) \\
 \frac{\neg \mathbf{b}, \mathbf{b}, \mathbf{p}, \neg \Diamond \mathbf{p}}{UnSat(\neg \mathbf{b}, \mathbf{b}, \dots)} (\Rightarrow)
 \end{array}$$

Notice that derivability in $\mathcal{T}_{\mathcal{A}}^R$ is conservative with respect to extensions of the background base $\mathcal{K}$. Accordingly, $\mathcal{T}_{\mathcal{A}}^R$ captures only the globally monotonic layer of conditional reasoning. To capture globally non-monotonic reasoning, we extend $\mathcal{T}_{\mathcal{A}}^R$ in the next section with a rule reflecting model minimality, whose application is not conservative under base extension.

4 Globally Non-Monotonic Layer

We now extend $\mathcal{T}_{\mathcal{A}}^R$ with a single rule capturing minimality of ranked models. The full calculus, denoted $\mathcal{T}_{\mathcal{A}}^{RMin}$, adds the following rule:

$$\frac{\Diamond A, \neg \Diamond B, \Delta \quad \& \quad \neg \Diamond A, \Gamma}{Sat(\{B\} \cup (\mathcal{K} \setminus \Gamma_{\mathcal{K}}^+))} \text{ (Min)}$$

where $\Gamma_{\mathcal{K}}^+ = \{(X \rightsquigarrow Y) \in \mathcal{K} \mid \Diamond X \in \Gamma\}$, and $\overline{(\mathcal{K} \setminus \Gamma_{\mathcal{K}}^+)}$ is the materialization of $(\mathcal{K} \setminus \Gamma_{\mathcal{K}}^+)$, i.e. $\overline{(\mathcal{K} \setminus \Gamma_{\mathcal{K}}^+)} = \{A \rightarrow B \mid (A \rightsquigarrow B) \in \mathcal{K} \setminus \Gamma_{\mathcal{K}}^+\}$. Crucially, extending the background base $\mathcal{K}$ with new conditionals may invalidate the concluding satisfiability check by introducing additional formulas into it. Consequently, derivability in $\mathcal{T}_{\mathcal{A}}^{RMin}$ is no longer conservative with respect to base extension, enabling the calculus to capture global non-monotonicity.

Notice that (Min) has the same numerator as the rule $(\mathcal{R}^\uparrow)$ and thus concerns tableaux representing worlds w_1 and w_2 , with $w_2 \prec w_1$ and $M, w_1 \not\models \Diamond B$. The concluding leaf checks whether B is classically compatible with the materialization of those conditionals whose antecedents have not already been placed below the rank of w_2 . Intuitively, if this check succeeds, a B -world could be placed below w_2 without violating $\mathcal{K}$; minimality therefore requires such a world to occur there, contradicting $\neg \Diamond B$. This fact is demonstrated formally in the following soundness proof.

¹ We denote a valuations by the string of atoms it satisfies, and the empty valuations by ϵ .

Theorem 4 (Soundness of $\mathcal{T}_{\mathcal{A}}^{RMin}$). *Every tableau derivable in $\mathcal{T}_{\mathcal{A}}^{RMin}$ relative to $\mathcal{K}$ is not satisfied by any minimal model of $\mathcal{K}$.*

Proof. The proof follows the same contrapositive argument as Theorem 2; all rules except (Min) are covered there. Consider an application of (Min) and suppose that its two numerator tableaux are satisfied in a minimal model M of $\mathcal{K}$. We show that its concluding *Sat* node cannot be correct. If the numerator tableaux are satisfied then there is a world w_2 in M , such that $\mathcal{R}^M(B) > \mathcal{R}^M(w_2)$ and $\mathcal{R}^M(X) < \mathcal{R}^M(w_2)$ for every $\Diamond X \in \Gamma$. Suppose, towards a contradiction, that all formulas $\{B\} \cup (\mathcal{K} \setminus \Gamma_{\mathcal{K}}^+)$ are satisfied by some valuation v . Add to M a world with valuation v at rank $\mathcal{R}^M(w_2)$. The resulting model is still a model of $\mathcal{K}$. Indeed, for each conditional in $\Gamma_{\mathcal{K}}^+$, the minimal worlds satisfying its antecedent already occur below rank $\mathcal{R}^M(w_2)$; every remaining conditional in $\mathcal{K}$ is respected by the new world because v satisfies its materialization. Thus we obtain a model M' of $\mathcal{K}$ such that $\mathcal{R}^{M'}(v) \leq \mathcal{R}^M(w_2) < \mathcal{R}^M(B) \leq \mathcal{R}^M(v)$. Hence the lowest rank of v can be decreased while preserving $\mathcal{K}$, contradicting the minimality of the valuation ranks of M .

Example 5. The fact $(\mathbf{b} \wedge \mathbf{g} \rightsquigarrow \mathbf{f})$ from Example 2 can be encoded by the following closed $\mathcal{T}_{\mathcal{A}}^{RMin}$ -derivation of $\{\mathbf{b} \wedge \mathbf{g}, \neg\Diamond(\mathbf{b} \wedge \mathbf{g}), \neg\mathbf{f}\}$ relative to $\mathcal{K}_{\Delta}$ (we omit unused formulas in tableaux for compact presentation):

$$\begin{array}{c}
 \frac{\mathbf{b} \wedge \mathbf{g}, \neg\Diamond(\mathbf{b} \wedge \mathbf{g}), \neg\mathbf{f}}{\frac{\neg\mathbf{b}, \mathbf{b} \wedge \mathbf{g}, \dots}{UnSat(\neg\mathbf{b}, \mathbf{b} \wedge \mathbf{g}, \dots)} (\Rightarrow) \quad \frac{\mathbf{b}, \neg\Diamond(\mathbf{b} \wedge \mathbf{g}), \neg\mathbf{f}, \dots}{\frac{\neg\Diamond\mathbf{b}, \mathbf{b}, \neg\mathbf{f}, \dots}{\mathbf{f}, \neg\Diamond\mathbf{b}, \mathbf{b}, \neg\mathbf{f}, \dots} (\rightsquigarrow) \quad \frac{\Diamond\mathbf{b}, \neg\Diamond(\mathbf{b} \wedge \mathbf{g}), \dots}{\mathbf{b}, \neg\Diamond\mathbf{b}} (\mathcal{R}^{\downarrow})} (\text{Min})} (\text{acut}) \\
 \frac{\mathbf{b}, \neg\Diamond(\mathbf{b} \wedge \mathbf{g}), \neg\mathbf{f}, \dots}{UnSat(\mathbf{f}, \neg\mathbf{f}, \dots)} (\Rightarrow) \quad \frac{\mathbf{b}, \neg\Diamond\mathbf{b}}{Sat(\mathbf{b} \wedge \mathbf{g}, \mathcal{K}_{\Delta})} (\text{Min})
 \end{array}$$

Since $\mathcal{T}_{\mathcal{A}}^{RMin}$ extends $\mathcal{T}_{\mathcal{A}}^R$ only by a concluding rule, the analyticity and termination arguments carry over unchanged. In particular, internal nodes in a derivation of Γ_0 still contain only formulas from $\mathcal{F}^*(\Gamma_0)$, while the new concluding satisfiability check may additionally contain materializations of conditionals from $\mathcal{K}$, still preserving analyticity.

Roadmap of the completeness proof. As in the globally monotonic case, completeness of $\mathcal{T}_{\mathcal{A}}^{RMin}$ is proved by reconstructing a countermodel from a saturated branch. Accommodating minimality, however, requires threading this construction through a three-way connection between the proof-theoretic, constructive, and semantic characterizations of rational closure. The proof proceeds in three main steps. At the first step, we select a specific target minimal model of $\mathcal{K}$ called “compatibility model” $Q_{\mathcal{K}}$ (Definition 6), whose layer k contains all valuations compatible with the k -th level of the base stratification $RSeq(\mathcal{K})$. This construction essentially repeats the “cumulative” model construction from [8] presenting it in possible-world representation. Using Theorem 1, we verify that $Q_{\mathcal{K}}$ is indeed a minimal model (Lemma 1). At the second step, we connect saturated derivations with the base stratification: by assigning structural ranks to tableaux on a saturated branch, the Prefix Lemma (Lemma 2) shows that every (subset-maximal) tableau faithfully encodes an element of some prefix of $RSeq(\mathcal{K})$. At the final step, we prove completeness of $\mathcal{T}_{\mathcal{A}}^{RMin}$ (Theorem 5) by matching every tableau on a saturated branch with a world on the corresponding layer of $Q_{\mathcal{K}}$ satisfying it.

Compatibility model. We will use one specific minimal ranked model of $\mathcal{K}$ as a countermodel for every non-derivable statement in $\mathcal{T}_{\mathcal{A}}^{RMin}$ w.r.t. $\mathcal{K}$. This model can be obtained from the stratification $RSeq(\mathcal{K})$ (see Definition 4) of $\mathcal{K}$ using the “cumulative” construction of Booth and Paris [8], represented here as a possible-world model. At each rank i it contains all valuations compatible with the conditionals C_i .

Definition 6. The compatibility model of $\mathcal{K}$ is defined as $Q_{\mathcal{K}} = \langle W_{\mathcal{K}}^Q, \preceq_{\mathcal{K}}^Q, V_{\mathcal{K}}^Q \rangle$, with worlds $W_{\mathcal{K}}^Q = \{ \langle i, v \rangle \mid i \in \mathbb{N}, v \text{ is a valuation, } v \models \overline{C_i} \}$; the valuation is $V_{\mathcal{K}}^Q(\langle i, v \rangle) = v$, and preference relation $\langle i, v \rangle \preceq_{\mathcal{K}}^Q \langle j, v' \rangle$ which holds iff $i \leq j$.

Thus, a valuation may occur at several layers, with its rank given by the first layer at which it is compatible with the corresponding materialization.

Example 6. In the compatibility model $Q_{\mathcal{K}_{\Delta}}$ of the base $\mathcal{K}_{\Delta}$ from Example 1 valuations ϵ , $\mathbf{f}$, and $\mathbf{bf}$ appear with all ranks; valuations $\mathbf{b}$ and $\mathbf{bp}$ appear with ranks 1 and above; valuations $\mathbf{p}$, $\mathbf{pf}$, and $\mathbf{bpf}$ appear with ranks 2 and above.

We can show that such a model is indeed minimal.

Lemma 1. $Q_{\mathcal{K}}$ is a minimal ranked model of $\mathcal{K}$.

Proof. The relation $\preceq_{\mathcal{K}}^Q$ is reflexive, transitive, total, and well-founded. Moreover, directly from the definitions, $\mathcal{R}^{Q_{\mathcal{K}}}(A) = \mathcal{R}_{\mathcal{K}}^{ex}(A)$ for every propositional formula A . Hence, for every $A \rightsquigarrow B \in \mathcal{K}$, either $\mathcal{R}_{\mathcal{K}}^{ex}(A) = \infty$, in which case $Q_{\mathcal{K}} \models A \rightsquigarrow B$ vacuously, or $\mathcal{R}^{Q_{\mathcal{K}}}(A) = \mathcal{R}_{\mathcal{K}}^{ex}(A) < \mathcal{R}_{\mathcal{K}}^{ex}(A \wedge \neg B) = \mathcal{R}^{Q_{\mathcal{K}}}(A \wedge \neg B)$, so $Q_{\mathcal{K}}$ is a model of $\mathcal{K}$. It remains to establish minimality. Let M be a minimal model of $\mathcal{K}$. Suppose towards a contradiction that $Q_{\mathcal{K}}$ is not minimal. Since M is minimal whereas $Q_{\mathcal{K}}$ is not, there is a valuation v such that $\mathcal{R}^M(v) < \mathcal{R}^{Q_{\mathcal{K}}}(v)$. Let $r = \mathcal{R}^M(v)$. Since v does not occur at layer r of $Q_{\mathcal{K}}$, it violates the materialization of C_r . Thus, for some $(A \rightsquigarrow B) \in C_r$, $v \models A \wedge \neg B$. Consequently, $\mathcal{R}^M(A) < \mathcal{R}^M(A \wedge \neg B) \leq r$. On the other hand, $(A \rightsquigarrow B) \in C_r$ implies $\mathcal{R}_{\mathcal{K}}^{ex}(A) \geq r$. By Theorem 1, $\mathcal{R}^M(A) = \mathcal{R}_{\mathcal{K}}^{ex}(A)$, yielding a contradiction.

Prefix Lemma. We next relate saturated branches in $\mathcal{T}_{\mathcal{A}}^{RMin}$ to prefixes of the constructive stratification $RSeq(\mathcal{K})$. As in the local completeness proof, we consider only subset-maximal tableaux on the branch. Each such tableau will be assigned a structural rank and shown to encode the corresponding layer of $RSeq(\mathcal{K})$. For a tableau Γ , let $N(\Gamma) = \{A \mid \neg \Diamond A \in \Gamma\}$, $P(\Gamma) = \{A \mid \Diamond A \in \Gamma\}$, and let $\Gamma_{\mathcal{K}}^- = \{A \rightsquigarrow B \in \mathcal{K} \mid A \in N(\Gamma)\}$. For subset-maximal tableaux Γ and Σ on a saturated branch $\mathcal{B}$, define $\Gamma \sqsupset_{\mathcal{B}} \Sigma$ iff $N(\Gamma) \subseteq N(\Sigma)$ and $P(\Gamma) \cap N(\Sigma) \neq \emptyset$. Intuitively, Σ represents a world strictly below Γ . Let $R^{\mathcal{B}}(\Gamma)$ be the maximal length of a $\sqsupset_{\mathcal{B}}$ -chain starting at Γ . This rank is well-defined because the branch is finite and $\sqsupset_{\mathcal{B}}$ is transitive and irreflexive.

Lemma 2 (Prefix Lemma). If Γ is a subset-maximal tableau on a saturated branch $\mathcal{B}$ of a derivation in $\mathcal{T}_{\mathcal{A}}^{RMin}$ then $\Gamma_{\mathcal{K}}^- = C_{R^{\mathcal{B}}(\Gamma)}$.

Proof. We proceed by strong induction on $r = R^{\mathcal{B}}(\Gamma)$. Suppose first that $r = 0$. For every antecedent A occurring in $\mathcal{K}$, saturation under $(acut)$ gives either $\Diamond A \in \Gamma$ or $\neg\Diamond A \in \Gamma$. The former, together with saturation under $(\mathcal{R}^\perp)$ and $(\leq)$, would yield a subset-maximal tableau Σ such that $\Gamma \sqsupset_{\mathcal{B}} \Sigma$, contradicting $r = 0$. Thus $\neg\Diamond A \in \Gamma$ for every $A \rightsquigarrow B \in \mathcal{K}$, and hence $\Gamma_{\mathcal{K}}^- = \mathcal{K} = C_0$. Now let $r > 0$, and choose Σ such that $\Gamma \sqsupset_{\mathcal{B}} \Sigma$ and $R^{\mathcal{B}}(\Sigma) = r - 1$. By the induction hypothesis, $\Sigma_{\mathcal{K}}^- = C_{r-1}$. Since $N(\Gamma) \subseteq N(\Sigma)$, we have $\Gamma_{\mathcal{K}}^- \subseteq \Sigma_{\mathcal{K}}^-$. Let $(A \rightsquigarrow B) \in \Gamma_{\mathcal{K}}^-$. Saturation under (Min) and $(acut)$, together with the fact that $\mathcal{B}$ cannot be closed by its concluding satisfiability check, implies that A is classically inconsistent with $\overline{\Sigma_{\mathcal{K}}^-}$. Hence A is exceptional for C_{r-1} , so $A \rightsquigarrow B \in C_r$. Conversely, let $A \rightsquigarrow B \in \Sigma_{\mathcal{K}}^- \setminus \Gamma_{\mathcal{K}}^-$. Saturation under $(acut)$ gives $\Diamond A \in \Gamma$, and saturation under $(\mathcal{R}^\perp)$ and $(\leq)$ yields a subset-maximal tableau Θ containing A and $\neg\Diamond A$, with $\Gamma \sqsupset_{\mathcal{B}} \Theta$. Thus $k = R^{\mathcal{B}}(\Theta) < r$, and by the induction hypothesis $\Theta_{\mathcal{K}}^- = C_k$. Because the branch cannot be closed by $(\Rightarrow)$, the propositional formulas in Θ are classically consistent. Moreover, saturation under $(acut)$ and $(\rightsquigarrow)$ ensures that for $(X \rightsquigarrow Y) \in \Theta_{\mathcal{K}}^-$ either $\neg X \in \Theta$ or $Y \in \Theta$. Since $A \in \Theta$, it follows that A is consistent with $\overline{C_k}$, and therefore also with $\overline{C_{r-1}}$. Hence A is not exceptional for C_{r-1} , so $A \rightsquigarrow B \notin C_r$. Therefore, $\Gamma_{\mathcal{K}}^- = C_r$.

Example 7. Consider the saturated branch $\mathcal{B}$ in the derivation in Example 3. It is easy to check that it also cannot be closed by (Min) , so the prefix lemma applies to it. For the two subset-maximal tableaux on it, $\Gamma_1 = \{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \Diamond\mathbf{b}\}$ and $\Gamma_2 = \{\mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}, \neg\mathbf{p}, \neg\Diamond\mathbf{p}\}$, we have $\Gamma_1 \sqsupset_{\mathcal{B}} \Gamma_2$, and $\Gamma_1^- = \{\mathbf{p} \rightsquigarrow \neg\mathbf{f}, \mathbf{p} \rightsquigarrow \mathbf{b}\} = C_1$ and $\Gamma_2^- = \{\mathbf{b} \rightsquigarrow \mathbf{f}, \mathbf{p} \rightsquigarrow \neg\mathbf{f}, \mathbf{p} \rightsquigarrow \mathbf{b}\} = C_0$.

Completeness of $\mathcal{T}_{\mathcal{A}}^{RMin}$ Using the prefix lemma, we can now satisfy every subset-maximal tableau on a saturated branch at the corresponding layer of the compatibility model $\mathcal{Q}_{\mathcal{K}}$.

Theorem 5 (Completeness of $\mathcal{T}_{\mathcal{A}}^{RMin}$). *Every tableau that is not satisfied in any minimal model of $\mathcal{K}$ is derivable in $\mathcal{T}_{\mathcal{A}}^{RMin}$ relative to $\mathcal{K}$.*

Proof. Suppose that Γ_0 has no closed derivation. Exhaustive rule application without repetitions yields a saturated branch $\mathcal{B}$. We show that every subset-maximal tableau Γ on $\mathcal{B}$ is satisfied in $\mathcal{Q}_{\mathcal{K}}$. Let $k = R^{\mathcal{B}}(\Gamma)$ and choose a valuation v_Γ satisfying all propositional formulas in Γ ; such a valuation exists since $\mathcal{B}$ cannot be closed by $(\Rightarrow)$. For every $X \rightsquigarrow Y \in \Gamma_{\mathcal{K}}^-$, saturation under $(acut)$ and $(\rightsquigarrow)$ ensures that either $\neg X \in \Gamma$ or $Y \in \Gamma$. Hence $v_\Gamma \models \overline{\Gamma_{\mathcal{K}}^-} = \overline{C_k}$, due to Lemma 2. Therefore, $w_\Gamma = \langle k, v_\Gamma \rangle$ is a world of $\mathcal{Q}_{\mathcal{K}}$. We prove by induction on k that w_Γ satisfies every formula in Γ . The claim is immediate for propositional formulas. Let $\neg\Diamond A \in \Gamma$. If $k = 0$, there are no worlds below w_Γ . Otherwise, choose a subset-maximal tableau Σ such that $\Gamma \sqsupset_{\mathcal{B}} \Sigma$ and $R^{\mathcal{B}}(\Sigma) = k - 1$. Since $\mathcal{B}$ is saturated under $(acut)$ and cannot be closed by (Min) , A is classically inconsistent with $\overline{\Sigma_{\mathcal{K}}^-} = \overline{C_{k-1}}$. As $C_j \supseteq C_{k-1}$ for every $j < k$, no world below layer k in $\mathcal{Q}_{\mathcal{K}}$ satisfies A . Thus $w_\Gamma \models \neg\Diamond A$. Finally, let $\Diamond A \in \Gamma$. Saturation under $(\mathcal{R}^\perp)$ and $(\leq)$ yields a subset-maximal tableau Θ containing A and $\neg\Diamond A$ such that $\Gamma \sqsupset_{\mathcal{B}} \Theta$. Hence $R^{\mathcal{B}}(\Theta) < k$, and by the induction hypothesis the world associated with Θ satisfies

A at a strictly lower layer than w_Γ . Therefore, $w_\Gamma \models \Diamond A$. Thus every subset-maximal tableau on $\mathcal{B}$ is satisfied in $\mathcal{Q}_\mathcal{K}$. Since Γ_0 is contained in some such tableau, it is also satisfied in this minimal model, contradicting the assumption of the theorem.

Example 8. Consider again the saturated branch $\mathcal{B}$ in the derivation in Example 3, which cannot be closed by (Min). The first subset-maximal tableau $\{\mathbf{p}, \neg\Diamond\mathbf{p}, \neg\mathbf{f}, \mathbf{b}, \Diamond\mathbf{b}\}$ on it is satisfied in $\mathcal{Q}_{\mathcal{K}_\Delta}$ in the world $\langle 1, \mathbf{pb} \rangle$, while the second subset-maximal tableau $\{\mathbf{b}, \neg\Diamond\mathbf{b}, \mathbf{f}, \neg\mathbf{p}, \neg\Diamond\mathbf{p}\}$ on it is satisfied in $\mathcal{Q}_{\mathcal{K}_\Delta}$ in the world $\langle 0, \mathbf{bf} \rangle$.

Soundness and completeness together with the equivalence between minimal entailment and rational closure imply that $\mathcal{T}_{\mathcal{A}}^{RMin}$ captures the latter.

Corollary 2. $(X \rightsquigarrow Y) \in RC(\mathcal{K})$ iff $\{X, \neg\Diamond X, \neg Y\}$ is derivable in $\mathcal{T}_{\mathcal{A}}^{RMin}$ relative to $\mathcal{K}$.

Notice that the derivation in Example 5 exhibits an apparent inefficiency: instead of decomposing the conjunction, it applies (*acut*) and creates a branch immediately closed by ($\Rightarrow$). This is a by-product of our two-stage design, which separates classical reasoning from modal and conditional reasoning, which we adopted to simplify the presentation. As Theorem 5 shows, analytic cuts nevertheless suffice to derive every non-satisfiable tableau. At the same time, the calculus may now be conservatively extended with classical rules, or with any other rules sound for minimal models, without affecting completeness. Such extensions may interleave classical and conditional reasoning or encode more complex reasoning steps, potentially leading to shorter and more readable derivations.

5 Proof Search and Complexity

In this section, we will analyse proof search for the two calculi, identify restricted yet complete rule-application strategies that yield polynomial bounds on derivation depth, and show that witnesses of non-entailment (i.e. saturated branches) can be reconstructed in optimal time complexity for both preferential entailment and rational closure.

We have already shown that proof search terminates whenever rule applications do not repeat tableaux. With some stronger restrictions we can also achieve a polynomial bound on the depth of proof search. Specifically, we call a rule application *greedy* if all tableaux in its numerator are subset-maximal on the current branch, and *redundant* if some tableau in its denominator is a subset of a tableau already occurring on that branch. We can show that this restriction does not violate derivability.

Proposition 1. Every derivation in $\mathcal{T}_{\mathcal{A}}^{RMin}$ can be transformed into one containing only greedy, non-redundant rule applications.

Proof. Each rule is monotone in its numerator tableaux: replacing one of them by a larger tableau only enlarges the tableaux in the denominator. This transformation also preserves correctness of concluding nodes, since the set tested by *UnSat* can only grow, whereas the set tested by *Sat* can only shrink. Therefore every rule can be replaced by the rule applied to subset-maximal extensions of tableaux in its numerator, while redundant applications may simply be omitted.

At the same time, the number of such restricted rule applications in every branch is bounded polynomially.

Proposition 2. *Every branch of a derivation of Γ in $\mathcal{T}_{\mathcal{A}}^{RMin}$ relative to $\mathcal{K}$ with only greedy and non-redundant rule applications has depth $O((|\Gamma| + |\mathcal{K}|)^2)$.*

Proof. Along any branch, every non-concluding rule other than $(\mathcal{R}^\downarrow)$ strictly enlarges one of its subset-maximal tableaux without increasing their number. Each such tableau contains at most $|\mathcal{F}^*(\Gamma)|$ formulas. Rule $(\mathcal{R}^\downarrow)$ introduces a new subset-maximal tableau and can be applied non-redundantly at most once for each relevant $\Diamond$ -formula, hence at most $|\mathcal{F}^*(\Gamma)|$ times. The branch depth is therefore quadratic in $|\mathcal{F}^*(\Gamma)|$, which itself is linear in the size of Γ and $\mathcal{K}$.

This bound yields an optimal certificate of non-derivability for $\mathcal{T}_{\mathcal{A}}^R$. A saturated branch can be guessed by selecting one conclusion of each branching rule and applying only greedy, non-redundant rules. Its depth is polynomial by Proposition 2, and we can verify in polynomial time that it cannot be closed by any concluding rule application, by guessing classical satisfying valuations for the concluding checks generated by $(\Rightarrow)$. Thus non-derivability in $\mathcal{T}_{\mathcal{A}}^R$ is in NP, and derivability is in coNP, matching the known coNP-completeness of preferential entailment [26].

For $\mathcal{T}_{\mathcal{A}}^{RMin}$, the same direct strategy would require checking that the satisfiability leaves introduced by (Min) are incorrect, i.e. solving classical unsatisfiability problems inside a nondeterministic computation, giving only a Σ_2^P upper bound. The optimal P^{NP} bound can instead be recovered using the subroutine that solves derivability in $\mathcal{T}_{\mathcal{A}}^{RMin}$ using exceptionality ranks and the compatibility model.

Lemma 3. *Given a base $\mathcal{K}$ and a tableau Γ , it can be decided in P^{NP} whether Γ is satisfied in $Q_{\mathcal{K}}$.*

Proof. Using polynomially many calls to an NP oracle, we construct the stratification $RSeq(\mathcal{K})$ and compute the rational closure ranks of all propositional formulas occurring under $\Diamond$ in Γ . It then suffices to inspect the polynomially many relevant layers k of $Q_{\mathcal{K}}$. At layer k , the modal formulas in Γ impose the conditions $\mathcal{R}_{\mathcal{K}}^{ex}(A) < k$ for every $\Diamond A \in \Gamma$, and $\mathcal{R}_{\mathcal{K}}^{ex}(A) \geq k$ for every $\neg\Diamond A \in \Gamma$, while the propositional part is satisfiable there iff $\overline{C_k} \cup (\Gamma \cap \mathcal{L}^P)$ is classically consistent. These conditions require only polynomially many additional NP-oracle calls.

The lemma provides a P^{NP} strategy for reconstructing a saturated branch in $\mathcal{T}_{\mathcal{A}}^{RMin}$: apply arbitrary greedy, non-redundant rules and, at every branching step, follow a child satisfied in $Q_{\mathcal{K}}$. By Proposition 2, only polynomially many such choices and concluding checks are required. Together with soundness and completeness, this yields a P^{NP} decision procedure for derivability in $\mathcal{T}_{\mathcal{A}}^{RMin}$, matching the known complexity of rational closure membership.

This strategy establishes the optimal upper bound but is not yet a practical proof-search procedure, since the guiding test essentially reconstructs the exceptionality stratification. Finding a natural complexity-optimal proof search strategy—for example, through a suitable rule ordering or stronger restrictions on rule applications—thus remains an interesting open problem.

6 Conclusion

We introduced the first analytic proof calculus for rational closure, modularly combining the globally monotonic reasoning of preferential entailment with globally non-monotonic reasoning based on model minimality. It derives valid entailments syntactically, while saturated branches in failed derivations witness non-entailment and support countermodel reconstruction. The completeness proof connects these proof-theoretic witnesses with exceptionality stratifications and minimal models. We also established polynomial bounds on branch depth and complexity-optimal proof-search procedures for both preferential entailment and rational closure. The framework provides a basis for proof-theoretic investigations of other rank-based entailment operators, including extensions to description logics, and suggests several directions for future work.

Use as explanations and certificates. The framework provides witnesses of both entailment and non-entailment that can serve as formal certificates and explanations by encoding semantic arguments. Their concrete use depends on the purpose and may require adjustments to the framework. For certificates, the central issue is the representation, which faces a trade-off between size and ease of verification: derivations in their current form require an external prover to check the leaves, or to establish non-closedness, but they can instead be supplied with certificates of classical (un)satisfiability, which may expand the derivation exponentially. For explanations, individual rules already expose useful information, e.g. ($\rightsquigarrow$) shows conditionals that contribute to a derivation and (Min) pinpoints where global non-monotonicity enters, but a systematic account of what can be extracted, and how, requires further study.

Proof complexity. Section 5 establishes polynomial bounds on branch depth, but the overall size of derivations remains largely unexplored. Although rational closure membership is P^{NP} -complete, there is no fundamental reason why derivations could not be polynomial in size, since they are not polynomially verifiable. Because our calculus combines both types of classical checks, it may be possible to represent the entire stratification-construction algorithm as a single derivation, either through a carefully chosen strategy of rule applications or by enriching the calculus with additional sound rules. Whether compact derivations can be obtained in this way is an open question.

Automation. The framework lends itself naturally to automation. Derivations can be found by direct proof search, while saturated branches can be encoded and searched for using provers such as ASP or QBF. Although the constructive characterization of rational closure already yields a simple and worst-case-optimal decision procedure, our calculus also provides a framework for investigating alternative approaches that may offer better average-case performance or improved efficiency in specific scenarios, particularly by combining locally and globally non-monotonic reasoning. This direction deserves a separate empirical study.

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