Graded Distributed Belief

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We introduce a new logic of graded distributed belief that allows us to express the fact that a coalition of agents distributively believe that a certain fact φ holds with at least strength k. We interpret our logic by means of computationally grounded semantics relying on the concept of belief base. The strength of the coalition's distributed belief is directly computed from the coalition's belief base after having merged its members' individual belief bases. We illustrate our logic with an intuitive example, formalizing the notion of epistemic disagreement. We also provide a sound and complete Hilbert-style axiomatization, decidability result obtained via filtration, and a tableaux-based decision procedure that allows us to state PSPACE-completeness for our logic.

1 Introduction

The idea of using belief bases as formal semantics for multi-agent epistemic logic was first introduced in [24] and further developed in [25, 26]. This approach aligns with the sentential (or syntactic) perspective on knowledge representation [20, 12, 31, 19], which holds that an agent's body of knowledge should be represented as a set of sentences in a formal language. The key novelty of belief base semantics, compared to traditional epistemic logic semantics based on multi-relational Kripke models [29, 11], lies in two main aspects. First, a possible world (or state) in a model is not treated as a primitive entity but is instead composed of the agents' belief bases and a valuation of propositional atoms. Second, the agents' accessibility relations are not explicitly part of the model but are determined *a posteriori* from their belief bases. Specifically, in this semantics, an agent at state S considers state S' possible (or epistemically accessible) if and only if S' satisfies all the formulas in the agent's belief base at S. This decomposition of a state into more fundamental elements is shared by various approaches in symbolic model checking and computationally grounded semantics for epistemic logic. These include frameworks based on interpreted systems, where a global state is decomposed into individual agents' local states [10, 23], as well as those that rely on the primitive notion of observability or visibility [16, 6, 13, 3]. At the language level, the belief base approach distinguishes explicit (or actual) belief from implicit (or potential) belief. The distinction between explicit and implicit belief has been widely discussed in the literature [22, 9]. In the belief base approach, this distinction is based on the concept of *deducibility*: explicit beliefs are those directly stored in an agent's belief base, while implicit beliefs consist of any information that can be logically inferred from those explicit beliefs.

In two subsequent works, the belief base approach has been shown to successfully represent notions of distributed and common belief [14, 27], as well as graded belief [28]. On the one hand, belief base semantics allows for a natural distinction between explicit and implicit distributed belief. While the explicit distributed belief of a coalition is given by the merging of the belief bases of its members, the implicit distributed belief corresponds to what can be deduced from the (collective) belief base resulting from this merging. On the other hand, the approach allows us to define a natural notion of the degree (or strength) of an agent's implicit belief that φ, understood as the maximum number of pieces of information

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that can be removed from the agent's belief base without preventing the agent from deducing φ from their explicit beliefs.

In this paper, we present a generalization of the belief base semantics for epistemic logic we introduced in [24, 26]. In the original semantics, agents' belief bases were simply sets of formulas built from a language including propositional facts and explicit beliefs: an agent's belief base could include both information about the world and information about other agents' belief bases. Our generalization moves from plain (ungraded) belief bases to graded belief bases by using a multiset representation. In a graded belief base, each piece of information is associated with a natural number representing the strength of the agent's explicit belief, with 0 associated to a formula α meaning that the agent has no explicit belief that α. We use this more general semantics to define a novel notion of *graded distributed belief*, as a piece of information φ that a coalition can deduce from their collective belief base with a given strength k. Given a graded belief base for each agent in a coalition, we compute the coalition's graded distributed belief in two steps. First, we merge the graded belief bases of the coalition's members to obtain a collective graded belief base. The degree that the coalition assigns to a formula α corresponds to the sum of the degrees that each member assigns to α. Second, we compute the coalition's degree of distributed belief in a certain fact φ as the amount of information that can be removed from the coalition's belief base without preventing it from deducing φ. We also show how our framework can be used to define a quantitative notion of of epistemic disagreement within a coalition, based on the amount of information that must be removed from the coalition's collective belief base to restore consistency. This notion of disagreement bears similarities to the measure of inconsistency defined in [18], namely, the minimal number of formulas that need to be removed from a belief base to restore consistency.

The paper is organized as follows. In Section 2, we first present the general framework: the graded belief base semantics, the modal language for representing implicit graded distributed belief, its semantic interpretation and an example illustrating our framework as well as the notion of epistemic disagreement within a coalition that can be defined in our framework. Following [26], in Section 3, we introduce an alternative Kripke-style semantics for our modal language, which serves as a technical tool for investigating the proof-theoretic aspects of our framework. Section 4 presents a Hilbert-style axiomatics for our logic of graded distributed belief. Then, in Section 5 we present a decision procedure based on tableaux, which allows us to establish PSPACE-completeness for our logic. Full proofs are provided in a technical annex at the end of the paper.

Before turning to the core of the paper, we briefly discuss some related work. Although the notion of graded distributed belief and the graded belief base semantics used to interpret it, as introduced in this paper, are new, the notion of plain distributed belief has been widely investigated in epistemic logic [11, 34, 1, 30, 7]. Moreover, the idea of having graded belief modalities for individual agents was explored in previous work [21], in line with work in ranking theory [32]. Other approaches employ graded modalities for individual agents, where the degree of belief is determined either by the number of worlds in which the believed formula holds true [17, 15, 5], or by the amount of evidence supporting it [2]. The notion of graded belief base is also used in possibility theory [8]. In the present paper, we generalize it to the multi-agent setting and to nested beliefs.

2 Framework

In this section, we present our graded belief base semantics and show how to use it to compute graded doxastic accessibility relations for agents and coalitions. Then, we introduce a modal language of graded distributed belief and interpret it using the semantics. We illustrate our language and semantics with the help of a concrete example.

2.1 Notation

In this paper we will work with graded sets (or multisets), where grade (or cardinality) of each element is either a natural number (including zero) or infinity (denoted ω). To avoid confusion we use notation $\mathbb{N}_0$ and $\mathbb{N}_1$ for natural numbers with and without zero respectively, and $\mathbb{N}_0^\omega$ and $\mathbb{N}_1^\omega$ for their extensions with element ω. We represent a graded set over set X by a function $f\colon X \to \mathbb{N}_0^\omega$, and define the *support* of this graded set as $Supp(f) = \{x \in X \mid f(x) > 0\}$. We denote the set of all multisets over X by $\mathscr{M}(X)$.

For $X \subseteq \mathbb{N}_0^\omega$ we will use notation $\min^* X = \min(X \cup \{0\})$ and $\max^* X = \max(X \cup \{\omega\})$ to avoid dealing with the case of empty set. We will also consider potentially infinite sums of grades. Since grades are natural numbers (or inifinity) such sums have natural well-behaved definition: we define it as sum of non-zero summands if there are finitely many such summands and none of them is ω, and as ω otherwise. Finally, we will use the notion of *partitions*, i.e. functions dividing the given grade $k \in \mathbb{N}_0$ among agents in coalition J: $\mathscr{P}art(J,k) = \{\delta : J \to \mathbb{N}_0 \mid \sum_{i \in J} \delta(i) = k\}$.

2.2 Semantics

We are going to present a belief base semantics for epistemic attitudes of agents that generalizes the belief base semantics introduced in [24, 26] to multisets. Multisets are used to represent agents' explicit beliefs with their strengths, weight or epistemic importance. Unlike the standard Kripke semantics for epistemic logic in which the notions of epistemic alternative and plausibility of a world (or state) are given as primitive, in this semantics they are defined from the primitive concept of graded belief base.

Assume a countably infinite set of atomic propositions Atm and a finite set of agents $Agt = \{1, \dots, n\}$. The set of non-empty coalitions is denoted by $2^{Agt*} = 2^{Agt} \setminus \{\emptyset\}$. We define the language $\mathscr{L}_0(Atm, Agt)$ for representing agents' graded explicit beliefs by the following grammar:

$$\alpha \quad ::= \quad p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \triangle_i^k \alpha,$$

where p ranges over Atm, i ranges over Agt and k ranges over $\mathbb{N}_1^\omega$. The formula $\triangle_i^k \alpha$ is read "agent i explicitly believes that α with at least degree k". For notational convenience, we abbreviate $\triangle_i \alpha \stackrel{def}{=} \triangle_i^1 \alpha$. The formula $\triangle_i \alpha$ is simply read "agent i explicitly believes that α".

For notational convenience we write $\mathscr{L}_0$ instead of $\mathscr{L}_0(Atm, Agt)$, when the context is unambiguous.

Definition 1. *A state is a tuple $S = (\mathscr{B}_1, \dots, \mathscr{B}_n, V)$ where for every $i \in Agt$, $\mathscr{B}_i \in \mathscr{M}(\mathscr{L}_0)$ is agent i's graded belief base, and $V \subseteq Atm$ is the actual environment. The set of all states is denoted by* $\mathbf{S}$.

Given a formula $\alpha \in \mathscr{L}_0$, $\mathscr{B}_i(\alpha)$ captures the strength of agent i's explicit belief that α.

The language $\mathscr{L}_0$ is interpreted with respect to states, as follows (we omit Boolean cases, as they are defined as usual).

Definition 2. *Let $S = (\mathscr{B}_1, \dots, \mathscr{B}_n, V) \in \mathbf{S}$. Then:*

$$S \models p \iff p \in V,$$
$$S \models \triangle_i^k \alpha \iff \mathscr{B}_i(\alpha) \geq k.$$

Observe in particular the following interpretation of the graded explicit belief operator: agent i explicitly believes that α with at least degree k if and only if the information α has an importance for the agent at least equal to k.

From the agents' graded belief bases $\mathscr{B}_1, \dots, \mathscr{B}_n$ it is natural to compute the collective graded belief base $\mathscr{B}_J \in \mathscr{M}(\mathscr{L}_0)$ of a coalition J: the degree of explicit belief of the coalition is equal to the sum of the degrees of beliefs of the coalition's members.

Definition 3. *Let $S = (\mathscr{B}_1, \dots, \mathscr{B}_n, V) \in \mathbf{S}$ and $J \in 2^{Agt*}$. Then, $\mathscr{B}_J(\alpha) = \sum_{i \in J} \mathscr{B}_i(\alpha)$ for every $\alpha \in \mathscr{L}_0$.*

The following definition introduces the notion of graded doxastic alternative for a coalition.

Definition 4. *Let $J \in 2^{Agt*}$ and let $k \in \mathbb{N}_0$. Then, $\mathscr{R}_J^k$ is the binary relation on the set $\mathbf{S}$ such that, for all $S = (\mathscr{B}_1, \dots, \mathscr{B}_n, V), S' = (\mathscr{B}'_1, \dots, \mathscr{B}'_n, V') \in \mathbf{S}$:*

$$S \mathscr{R}_J^k S' \text{ if and only if } \sum_{\substack{\alpha \in \mathscr{L}_0 \\ S' \not\models \alpha}} \mathscr{B}_J(\alpha) \leq k.$$

$S \mathscr{R}_J^k S'$ means that, from the point of view of the coalition J at state S, state S' is at most k-implausible, where the degree of implausibility of a state for a coalition is equal to the weighted sum of the coalition's explicit beliefs that are not satisfied at the state. This means that the degree of implausibility of a state for a coalition depends on i) how much information that the coalition has in its belief base is not satisfied at the state, and ii) how important is that information for the coalition.

Notice that $S \mathscr{R}_J^k S'$ can also be interpreted as the fact that state S' is considered possible for the coalition J after removing from its collective belief base a body of information of importance at most equal to k. Indeed, $\sum_{\substack{\alpha \in \mathscr{L}_0 \\ S' \not\models \alpha}} \mathscr{B}_J(\alpha)$ can also be conceived as the total amount of importance for the coalition J at state S of the information that is not satisfied at state S'.

A graded doxastic accessibility relation $\mathscr{R}_J^k$ induces a plausibility ordering over states, as in [32, 21]. For notational convenience, we write $\mathscr{R}_J$ instead of $\mathscr{R}_J^0$. Clearly, $S \mathscr{R}_J S'$ if and only iff $\forall \alpha \in \mathscr{L}_0$, if $\mathscr{B}_J(\alpha) > 0$ then $S' \models \alpha$. In words, a state is 0-implausible from the point of view of a coalition if it satisfies all information in the coalition's belief base.

Before concluding this section, we define the concept of a model as a state supplemented with a set of states, called *context*. The latter includes all states compatible with the the agents' common ground [33], i.e., the body of information that the agents commonly believe to be the case.

Definition 5. *A multi-agent graded belief model (MAGBM) is a pair (S, U), where $S \in \mathbf{S}$ and $U \subseteq \mathbf{S}$. The class of models is denoted by $\mathbf{M}$.*

2.3 Language

We consider a modal language $\mathscr{L}(Atm, Agt)$ that extends the language $\mathscr{L}_0(Atm, Agt)$ given above with graded distributed belief modalities. It is is defined by the following grammar:

$$\varphi \quad ::= \quad \alpha \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box_J^k \varphi,$$

where α ranges over $\mathscr{L}_0(Atm, Agt)$, J ranges over 2^{Agt*} and k ranges over $\mathbb{N}_0$. For notational convenience we write and $\mathscr{L}$ instead of $\mathscr{L}(Atm, Agt)$, when the context is unambiguous. The other Boolean constructions $\top$, $\bot$, $\rightarrow$ and $\leftrightarrow$ are defined in the standard way.

We interpret the modal language $\mathscr{L}$ relative to a model by means of the graded accessibility relations $\mathscr{R}_J^k$. (We omit the Boolean cases, as they are defined in the usual way.)

Definition 6. *Let $(S, U) \in \mathbf{M}$. Then:*

$$(S, U) \models \alpha \iff S \models \alpha,$$
$$(S, U) \models \Box_J^k \varphi \iff \forall S' \in U, \text{ if } S \mathscr{R}_J^k S' \text{ then } (S', U) \models \varphi.$$

The modal formula $\Box_J^k \varphi$ is read "coalition J would implicitly believe that φ, for every removal from its belief base of a body of information of importance at most equal to k". The value k can also be conceived as the extent to which coalition J distributively believes that φ. Indeed, the higher the importance of the information that can be removed from the coalition's belief base without affecting what the coalition can infer, the stronger the inference and so the coalition's resulting distributed belief. Thus, $\Box_J^k \varphi$ can also be read "coalition J has an implicit distributed belief that φ of degree (or strength) at least k". The abbreviation $\Diamond_J^k \varphi \stackrel{\text{def}}{=} \neg \Box_J^k \neg \varphi$ defines the concept of distributed belief compatibility. The formula $\Diamond_J^k \varphi$ has to be read "φ would be compatible with coalition J's explicit beliefs, for some removal from its collective belief base of a body of information of importance at most equal to k".

2.4 Conceptual Analysis

We are going to show how our language and semantics can be leveraged to formally represent graded distributed belief as well as degree of epistemic disagreement within a coalition of agents. The latter notion is formally defined by the following abbreviation: $Disagree(J,k) \stackrel{def}{=} \Box_J^{k-1} \bot$ for $k \geq 1$. $Disagree(J,k)$ means that within the coalition J there is an epistemic disagreement of at least strength k.

Example 1. *Ann, Bob, Cath and John are the four members of a research project evaluation committee. Their task is to decide whether a submitted project for funding can be included in the list of "fundable" projects or not. They are all convinced with at least strength $k_0 > 0$ that a project should be included in the list (in) if and only if its idea is innovative (id) and, at the same time, the project's consortium is of high scientific standard (hi). This hypothesis is captured by the following abbreviation:*

$$\alpha_1 \stackrel{def}{=} \bigwedge_{i \in \{Ann, Bob, Cath, John\}} \triangle_i^{k_0} \big(in \leftrightarrow (id \wedge hi)\big).$$

However, they have diverging opinions and, in some cases, have not yet formed an opinion regarding these qualities of the project. In particular, Ann explicitly believes with degree $k_1 > 0$ that the project's idea is innovative, and Cath believes the opposite with degree $k_3 > 0$, Bob explicitly believes with degree $k_2 > 0$ that the project's consortium is of high scientific standard, and John explicitly believes the opposite with degree $k_4 > 0$. This hypothesis is captured by the following abbreviation:

$$\alpha_2 \stackrel{def}{=} \triangle_{Ann}^{k_1} id \wedge \triangle_{Bob}^{k_2} hi \wedge \triangle_{Cath}^{k_3} \neg id \wedge \triangle_{John}^{k_4} \neg hi.$$

It is routine to verify that coalition $\{Ann, Bob\}$ implicitly believes with degree $(\min\{2k_0, k_1, k_2\} - 1)$ that the project should be included in the list, while coalition $\{Cath, John\}$ believes the opposite with degree $(\min\{2k_0, k_3 + k_4\} - 1)$:

$$\models (\alpha_1 \wedge \alpha_2) \ \rightarrow \ (\Box_{\{Ann,Bob\}}^{\min\{2k_0,k_1,k_2\}-1} in \ \wedge \ \Box_{\{Cath,John\}}^{\min\{2k_0,k_3+k_4\}-1} \neg in).$$

Moreover, when the explicit information is restricted to $\alpha_1 \wedge \alpha_2$ there is no disagreement within these coalitions (i.e. there exist models where $\alpha_1 \wedge \alpha_2$ is true and $Disagree(J,1)$ is false), while all four agents together have an epistemic disagreement of at least strength $(\min\{k_1, k_3\} + \min\{k_2, k_4\})$:

$$\not\models (\alpha_1 \wedge \alpha_2) \ \rightarrow \ (Disagree(\{Ann, Bob\}, 1) \vee Disagree(\{Cath, John\}, 1))$$
$$\models (\alpha_1 \wedge \alpha_2) \ \rightarrow \ Disagree(\{Ann, Bob, Cath, John\}, \min\{k_1, k_3\} + \min\{k_2, k_4\}).$$

3 Alternative semantics

To explore the proposed logic we will follow the general approach of [24] and introduce two alternative equivalent semantical characterizations: *notional* and *quasi-notional graded doxastic models*.

First, we redefine models in terms closer to Kripke semantics for modal logics by introducing the notion of distance between states for a given coalition of agents as the sum of degrees in the first world of all beliefs not satisfied in the second world for all agents in the coalition. Note that this notion of distance is not (in general) symmetric.

Definition 7. *A* notional graded doxastic model (NGDM) *is a tuple* $M = (W, \mathscr{D}, \rho, \mathscr{V})$ *where* W *is a set of worlds,* $\mathscr{D} : Agt \times W \to \mathscr{M}(\mathscr{L}_0)$ *is a doxastic function,* $\rho : 2^{Agt*} \times W \times W \to \mathbb{N}_0^{\omega}$ *is a distance function, and* $\mathscr{V} : \mathbb{P} \to 2^W$ *is a valuation, such that:*

$$\rho(J,w,u) = \sum_{\substack{\alpha \in \mathscr{L}_0 \\ (M,u) \not\models \alpha}} \sum_{i \in J} \mathscr{D}(i,w)(\alpha) \qquad \text{(NGDM-DOX)}$$

with satisfaction relation defined as follows (omitting Boolean cases, defined in the usual way):

$(M,w) \models p \quad \Leftrightarrow \quad w \in \mathscr{V}(p)$

$(M,w) \models \triangle_i^k \alpha \quad \Leftrightarrow \quad \mathscr{D}(i,w)(\alpha) \geq k$

$(M,w) \models \Box_J^k \varphi \quad \Leftrightarrow \quad \forall u \in W : \rho(J,w,u) \leq k \Rightarrow (M,u) \models \varphi$

M is called finite *when W is finite and* $Supp(\mathscr{D}(i,w))$ *is finite for every i and w.*

Notice that condition (NGDM-DOX) fully defines distance function ρ via doxastic function $\mathscr{D}$ and the resulting distance function is *additive on coalitions*:

$$\rho(J,w,u) = \sum_{i \in J} \rho(\{i\},w,u) \qquad \text{(NGDM-}\rho\text{-ADD)}$$

However, condition (NGDM-DOX) itself is not axiomatizable (as follows from Lem. 2 below which provides a model transformation obtaining (NGDM-DOX)), therefore we also introduce the notion of *quasi-notional graded doxastic models (QNGDMs)* where this condition is weakened to become axiomatizable. First, instead of equality, we require that the distance is no smaller than the sum of degres of the unsatisfied beliefs (condition (QNGDM-DOX)). Additionally, we require a weakened version of (NGDM-ρ-ADD) (condition (QNGDM-ρ-ADD)): the finite distance for every coalition J can be partitioned into summands $\delta(i)$ for every agent $i \in J$ such that $\delta(i)$ is at least the distance for any agent $i \in J$, and moreover the distance for any sub-coalition does not exceed the sum of $\delta(i)$ for the agents involved.

Definition 8. *A* quasi-notional graded doxastic model (QNGDM) *is a tuple* $M = (W, \mathscr{D}, \rho, \mathscr{V})$ *where* W, ρ, $\mathscr{D}$, $\mathscr{V}$ *are as in Def. 7 except that (NGDM-DOX) is replaced by the two following weaker conditions for every* $J \in 2^{Agt*}$ *and* $w, u \in W$ *if* $\rho(J,w,u) \neq \omega$*:*

$$\rho(J,w,u) \geq \sum_{\substack{\alpha \in \mathscr{L}_0 \\ (M,u) \not\models \alpha}} \sum_{i \in J} \mathscr{D}(i,w)(\alpha) \qquad \text{(QNGDM-DOX)}$$

$$\exists \delta \in \mathscr{P}art(\rho(J,w,u),J) \text{ such that for any non-empty } J' \subset J, \sum_{i \in J'} \delta(i) \geq \rho(J',w,u) \qquad \text{(QNGDM-}\rho\text{-ADD)}$$

We will now show equivalence of the three semantics following the strategy of [26]: we will first show that semantics of QNGDMs satisfies the finite model property (Lem. 1), then we will show how every satisfying *finite* QNGDM can be transformed into a satisfying NGDM (Lem. 2), satisfying NGDM — into a satisfying MAGBM (Lem. 3), and a satisfying MAGBM into a satisfying QNGDM (Lem. 4), closing the circle.

To establish Finite Model Property for QNGDMs we adapt the standard filtration technique.

Lemma 1. *If M is QNGDM satisfying $\varphi \in \mathcal{L}$ then there exists a finite QNGDM M' satisfying φ.*

Proof. (Sketch) We consider an equivalence relation $\equiv_\varphi$ on worlds of M, relating worlds with the same evaluation on all subformulas of φ. Then we transform $M = (W, \rho, \mathscr{D}, \mathscr{V})$ into finite a QNGDM $(W', \rho', \mathscr{D}', \mathscr{V}')$ as follows: $W' = {}^{W}\!/\!_{\equiv_\varphi}$; $\rho'(J,U,V) \overset{def}{=} \min\{\rho(J,u,v) \mid u \in U, v \in V\}$; $\mathscr{D}'(i,U)(\alpha) \overset{def}{=} \max^*\{k \mid \triangle_i^k \alpha$ *is a subformula of* φ *and* $\mathscr{D}(i,u)(\alpha) \geq k$ *for all* $u \in U\}$; $\mathscr{V}'(p) \overset{def}{=} \{U \mid U \subseteq \mathscr{V}(p)\}$. We check that M' preserves evaluation on subformulas of φ and satisfies both conditions of QNGDMs. □

Now we show how to transform a finite QNGDM into a (finite) NGDM. We will adapt the idea of two-stage model tranformation for distributed belief bases from [14] to our graded setting. At the fist stage we achieve (NGDM-ρ-ADD) in a QNDM by creating copies of each world for every possible coalition and redefining distances for them on the basis of partitions given by condition (QNGDM-ρ-ADD). At the second stage we achieve (NGDM-DOX) by adapting the transformation from [28] to our case: introducing a fresh characterizing atom for each world and add such atoms as beliefs with the required degree to satisfy the equality in (NGDM-DOX).

Lemma 2. *If M is a finite QNGDM satisfying $\varphi \in \mathcal{L}$ then there exists a finite NGDM M'' satisfying φ.*

Proof. (Sketch) We first change the set of worlds W in M to $W' = W \times 2^{Agt*}$, keep $\mathscr{V}$ and $\mathscr{D}$ the same for each copy, and redefine distances for copies using δ from condition (QNGDM-ρ-ADD) for J, w and u: $\rho'(J',(w,J''),(u,J)) = \sum_{i \in J'} \delta(i)$ for $J' \subseteq J$ and $\rho'(J',(w,J''),(u,J)) = \omega$ otherwise. With such definition of distances, the transformed model M' preserves the satisfaction relation for each copy, trivially satisfies (NGDM-ρ-ADD), and still satisfies (QNGDM-DOX), since distances between copies did not decrease w.r.t. original distances (by condition (QNGDM-ρ-ADD)). At the second stage, for each $w' \in W'$ we select a distinctive atom $\chi(w')$ not appearing in φ and in any $\alpha \in Supp(\mathscr{D}'(i,w'))$ in M' (which we can do since M' is finite), and change $\mathscr{V}'$ to $\mathscr{V}''$ such that $\mathscr{V}''(\chi(w')) = W' \setminus \{w'\}$ (not changing the valuation on other atoms). Then we change the degrees of these atoms: $\mathscr{D}'(i,w)(\chi(u)) = \rho(i,w,u) - \sum_{\substack{\alpha \in \mathscr{L}_0 \\ (M',u) \not\models \alpha}} \mathscr{D}(i,w)(\alpha)$, turning inequality in (QNGDM-DOX) into equality for $M'' = (W', \rho', \mathscr{D}'', \mathscr{V}'')$. □

We can easily transform a satisfying NGDM into a satisfying MAGBM.

Lemma 3. *If φ is satisfied by some NGDM then φ is satisfied by some MAGBM.*

Proof. (Sketch) Each world can be mapped into a state (by reconstructing belief bases from the doxastic function) and the evaluation will be preserved thanks to the condition (NGDM-DOX). □

And, finally, we can move from MAGBMs back to QNGDMs.

Lemma 4. *If φ is satisfied by some MAGBM then φ is satisfied by some QNGDM.*

Proof. (Sketch) The doxastic function is defined by belief bases, and the distances can be defined via condition (NGDM-DOX), apart from the distances to the initial state S defined as ω (to reflect that, in general, S does not belong to the context U). □

Thus, we have established the equivalence of all three introduced semantics (MAGBMs, NGDMs, and QNGDMs), which will be crucial for axiomatization of the proposed logic. Moreover, the described transformations turn an arbitrary QNGDM into an NGDM of exponential size, which implies decidability of the proposed logic (since all NGDMs of bounded size can be checked for satisfaction in finite time).

4 Axiomatics

In this section, we present an axiomatization for the proposed logic **LGDDA** (Logic of Graded Distributed Doxastic Attitudes) based on the semantics of QNGDMs introduced in the previous section.

Definition 9. *Logic **LGDDA** extends the classical propositional logic by the following axioms and rules:*

$$\frac{\varphi}{\Box_J^k \varphi} \qquad (\mathbf{Nec}_{\Box_J^k})$$

$$\Box_J^k(\varphi \to \psi) \to (\Box_J^k \varphi \to \Box_J^k \psi) \qquad (\mathbf{K}_{\Box_J^k})$$

$$\triangle_i^k \alpha \to \triangle_i^{k'} \alpha \qquad \textit{if } k \geq k' \qquad (\mathbf{Mon}_{\triangle_i^l})$$

$$\Big(\bigwedge_{\triangle_i^{k'} \alpha \in \Omega} \triangle_i^{k'} \alpha\Big) \to \Box_J^k \bigvee_{\substack{\Omega' \subseteq \Omega \\ Sum(\Omega') \leq k}} \bigwedge_{\triangle_i^{k'} \alpha \in \Omega \setminus \Omega'} \alpha \qquad \begin{array}{l} \textit{if } \Omega \subseteq \{\triangle_i^{k'} \alpha \mid i \in J\} \\ \textit{and } \triangle_i^{k'} \alpha, \triangle_i^{k''} \alpha \in \Omega \Rightarrow k' = k'' \\ \textit{where } Sum(\Omega') = \sum_{\triangle_i^{k'} \alpha \in \Omega'} k' \end{array} \qquad (\mathbf{Int}_{\triangle_i^{k'}, \Box_J^k})$$

$$\Big(\bigwedge_{\Box_{J'}^{k'} \psi \in \Psi} \Box_{J'}^{k'} \psi\Big) \to \Box_J^k \bigvee_{\delta \in \mathscr{P}art(k,J)} \bigwedge_{\substack{\Box_{J'}^{k'} \psi \in \Psi \\ \sum_{i \in J'} \delta(i) \leq k'}} \psi \qquad \textit{if } \Psi \subseteq \{\Box_{J'}^{k'} \psi \mid J' \subseteq J\} \qquad (\mathbf{Int}_{\Box_{J'}^{k'}, \Box_J^k})$$

*$\varphi \in \mathscr{L}$ is said to be derivable from $\Gamma \subseteq \mathscr{L}$ in **LGDDA** (denoted $\Gamma \vdash_{\mathbf{LGDDA}} \varphi$) when there is finite $\Gamma_f \subseteq \Gamma$ such that formula $(\bigwedge_{\psi \in \Gamma_f} \psi) \to \varphi$ can be derived using axioms and rules of **LGDDA**.*

Rule ($\mathbf{Nec}_{\Box_J^k}$) and axiom ($\mathbf{K}_{\Box_J^k}$) reflect the fact that $\Box_J^k$ is a normal modality. Monotonicity axiom ($\mathbf{Mon}_{\triangle_i^l}$) reflects the fact that k in modality $\triangle_i^k$ gives a lower bound on the weight of the belief. Two final axioms determine the interaction between triangles and boxes, and between boxes with different coalitions and degrees respectively. Informally, axiom ($\mathbf{Int}_{\triangle_i^{k'}, \Box_J^k}$) states that coalition J believes with level k that their pulled beliefs are correct apart from some subset with the cummulative importance not exceeding k. Axiom ($\mathbf{Int}_{\Box_{J'}^{k'}, \Box_J^k}$) captures the fact that the distance d between states (as defined by (NGDM-DOX)) for coalition J can be partitioned into distances $\delta(i)$ for $i \in J$ such that any graded belief for a subcoalition J' with degree k' is preserved for coalition J with degree $k \geq d$, as long as k' is greater than the sum of distances $\delta(i)$ for $i \in J'$. Notice that the following two validities, defining monotonicity of boxes w.r.t. coalition and degrees, are instances of ($\mathbf{Int}_{\Box_{J'}^{k'}, \Box_J^k}$) for the case when there is only one box on the left:

$$\Box_J^k \varphi \to \Box_J^{k'} \varphi \qquad \textit{if } k \geq k' \qquad (\mathbf{Mon}^k_{\Box_J^k})$$

$$\Box_J^k \varphi \to \Box_{J'}^k \varphi \qquad \textit{if } J \subseteq J' \qquad (\mathbf{Mon}^J_{\Box_J^k})$$

Remark 1. *Note that the axiomatization in Def. 9 does not simply adjust and merge axioms for graded beliefs and distributed beliefs. While ($\mathbf{Int}_{\triangle_i^{k'}, \Box_J^k}$) is a natural adjustment of axiom ($\mathbf{Int}_{\Delta_i, \Box_i}$) from [28] for the case of graded belief bases, the logic of distributed belief from [14] requires only axiom ($\mathbf{Mon}^J_{\Box_J^k}$), and our logic for distributed graded belief requires significantly more sophisticated (and perhaps less intuitive) axiom ($\mathbf{Int}_{\Box_{J'}^{k'}, \Box_J^k}$) that reflects the combinatorics of partition of distances for coalitions and subcoalitions captured in condition* (QNGDM-ρ-ADD).

The proposed axiomatization is sound and complete w.r.t. QNGDM model semantics. To prove completeness we construct a canonical QNGDM, with rule ($\mathbf{Nec}_{\Box_J^k}$) and axioms ($\mathbf{K}_{\Box_J^k}$) and ($\mathbf{Mon}_{\triangle_i^l}$) used to establish the truth lemma, and axioms ($\mathbf{Int}_{\triangle_i^{k'}, \Box_J^k}$) and ($\mathbf{Int}_{\Box_{J'}^{k'}, \Box_J^k}$) characterizing conditions (QNGDM-DOX) and (QNGDM-ρ-ADD) of QNGDMs respectively.

Theorem 1. $\Gamma \vdash_{\boldsymbol{LGDDA}} \varphi$ *iff* $\Gamma \vDash_{QNGDM} \varphi$ *(i.e. any pointed QDNM satisfying all formulas from* Γ *also satisfies* φ*).*

Proof. *(Sketch)* **Soundness:** Axiom ($\mathbf{Int}_{\triangle_i^{k'},\Box_J^k}$) is valid in any QNGDM due to condition (QNGDM-DOX) (if $\rho(J,w,u) \leq k$ then the disjunct for $\Omega' = \{\triangle_i^{k'}\alpha \in \Omega \mid M,u \not\models \alpha\}$ will be satisfied in u) and axiom ($\mathbf{Int}_{\triangle_i^{k'},\Box_J^k}$) — due to condition (QNGDM-ρ-ADD) (it ensures existence of partition δ that corresponds to a satisfied disjunct), validity of other axioms and preservation of validity by the rules is trivial.

Completeness: We construct a canonical model $M^C = (W^C, \mathscr{D}^C, \rho^C, \mathscr{V}^C)$, where W^C contains **LGDDA**-maxiconsistent sets of formulas (i.e. maximal not deriving $\perp$), $\mathscr{D}^C(i,\Phi)(\alpha) = \max^*\{l\colon \triangle_i^l\alpha \in \Phi\}$; $\rho^C(J,\Phi,\Phi') = \min^*\{k\colon \Box_J^k\varphi \in \Phi \Rightarrow \varphi \in \Phi'\}$; $\mathscr{V}^C(p) = \{\Phi \in W^C\colon p \in \Phi\}$. The truth lemma stating $M^C,\Phi \models \psi \Leftrightarrow \psi \in \Phi$ is proved by structural induction on ψ, axiom ($\mathbf{Mon}_{\triangle_i^l}$) is used for the case of triangles, and the case of box follows from presence of ($\mathbf{K}_{\Box_J^k}$) and ($\mathbf{Nec}_{\Box_J^k}$) as usual. Then conditions (QNGDM-DOX) and (QNGDM-ρ-ADD) can be proved contrapositively using axioms ($\mathbf{Int}_{\triangle_i^{k'},\Box_J^k}$) and ($\mathbf{Int}_{\Box_{J'}^{k'},\Box_J^k}$) respectively. Failure of (QNGDM-DOX) for some w, u, and J would allow us to select some finite set Ω of beliefs false in u with aggregated weight in w greater than $\rho(J,w,u)$, and applying axiom ($\mathbf{Int}_{\triangle_i^{k'},\Box_J^k}$) we would conclude that some beliefs in Ω are true in u. Failure of (QNGDM-ρ-ADD) for some w, u, and J would imply existence of subcoalition $J_\delta \subset J$ for every $\delta \in \mathscr{P}art(\rho^C(J,w,u),J)$ such that $d_\delta = \sum_{i \in J_\delta} \delta(i) < \rho^C(J_\delta,\Phi,\Phi')$, which also implies existence of $\psi_\delta \in \mathscr{L}$ such that $\Box_{J_\delta}^{d_\delta}\psi_\delta$ is satisfied in w but ψ_δ is not satisfied in u. Taking $\Omega = \{\Box_{J_\delta}^{d_\delta}\psi_\delta \mid \delta \in \mathscr{P}art(\rho^C(J,w,u),J)\}$ and applying axiom ($\mathbf{Int}_{\triangle_i^{k'},\Box_J^k}$) to it, we can derive that some ψ_δ is satisfied in u, reaching contradiction. □

5 Tableau calculus and satisfiability checking

In this section, we present a tableau-based decision procedure for our logic and establish PSPACE-completeness of satisfiability checking (the same complexity as the basic logic of belief bases in [26]).

Definition 10. *The tableau calculus* $\boldsymbol{Tab_{LGDDA}}$ *extends the standard tableau calculus* $\boldsymbol{LK}$[1] *for the classical logic with the following two rules:*

$$\frac{\{\triangle_i^k\alpha, \neg\triangle_i^{k+t}\alpha\} \cup X}{\{\perp\}}\ (\triangle\text{-}Mon) \qquad \frac{\{\neg\Box_J^k\varphi\} \cup X}{\{\neg\varphi\} \cup Y_1^{\Box\downarrow} \cup Y_1^{\Delta_J\downarrow} \mid \ldots \mid \{\neg\varphi\} \cup Y_N^{\Box\downarrow} \cup Y_N^{\Delta_J\downarrow}}\ (\Box\text{-}Elim)$$

$$\begin{aligned}
\textit{where} \qquad \{Y_1,\ldots,Y_N\} &= \{Y \subseteq X \mid \exists\delta \in \mathscr{P}art(k,J):\ \ \Box_{J'}^{k'}\psi \in Y \Rightarrow \textstyle\sum_{i\in J'}\delta(i) \leq k' \textit{ and} \\
&\qquad\qquad \mathscr{D}_{agg}(i, X\setminus Y) \leq \delta(i)\ \forall i \in J\} \\
\mathscr{D}_{agg}(i, X\setminus Y) &= \textstyle\sum_{\alpha\in\mathscr{L}_0} \max^*\{k'\colon \triangle_i^{k'}\alpha \in X \setminus Y\} \\
Y^{\Box\downarrow} \cup Y^{\Delta_J\downarrow} &= \{\psi \mid \Box_{J'}^{k'}\psi \in Y\} \cup \{\alpha \mid \triangle_i^{k'}\alpha \in Y, i \in J\}
\end{aligned}$$

Each tableau rule states that the satisfiability of a set of formulas above the line (called *numerator*) implies satisfiability of at least one of the sets of formulas below the line separated by the symbol '|' (called *denominators*). We can use it to derive the non-satisfiability of some formula by applying the rules sequentially, with each branch ending with a set containing $\perp$ (thus unsatisfiable). Such tree-like derivations are called *closed tableaux*. For our logic, it is sufficient to add just one rule for each modality (in addition to standard rules for the classical connectives). The rule *(△-Mon)* captures the monotonicity of triangles w.r.t. grades, and the rule *(□-Elim)* adapts the rule eliminating negative boxes

[1]The formal definition of analytic tableaux and the calculus **LK** is in included in the Appendix.

to our logic: if $\{\neg\Box_J^k\varphi\}\cup X$ is satisfied in some world w of some model M then there should exist a world u with $\rho(J,w,u)\leq k$ and a partition $\delta\in\mathscr{P}art(k,J)$, such that u satisfies $\neg\varphi$, all boxed formulas for subcoalitions with smaller distance, and some selection of triangled formulas in X, such that the cumulative degree of the rest of triangled formulas does not exceed k. Note that the rule *($\Box$-Elim)* has potentially exponentially many denominators. At the same time, in all rules (apart from the closing ones) the denominators contain only subformulas of formulas in the numerator, and the total number of connectives and modalities in each denominator decreases w.r.t. the numerator, so the length of each branch is linear w.r.t. size of the initial formula. The resulting calculus is sound and complete w.r.t. logic **LGDDA**.

Theorem 2. *A closed tableau in* $\textbf{Tab}_{\textbf{LGDDA}}$ *starting from* $\{\varphi\}$ *exists iff* φ *is not satisfiable in* ***LGDDA***.

Proof. (Sketch) The soundness of all rules w.r.t. QNGDM semantics is straightforward. To prove completeness we show that for each set Γ underivable in $\textbf{Tab}_{\textbf{LGDDA}}$ (i.e. there are no closed tableaux starting from Γ) there is a pointed QNGDM model (M,w_0) satisfying all the formulas in Γ. We do it by induction on the number of connectives and modalities in Γ. First, if Γ contains formulas with double negation, conjunction or negated conjunction on the top level, they can be decomposed according to the rules and the statement follows straightforwardly from the inductive hypotheses. Otherwise Γ has only (possibly negated) atoms, $\triangle$-formulas and $\Box$-formulas. We start a model construction with one world w_0 which satisfy exactly atoms in $(\mathbb{P}\cap\Gamma)$ and with doxastic function $\mathscr{D}(i,w_0)(\alpha)=\max^*\{k':\triangle_i^{k'}\alpha\in\Gamma\}$. It is easy to check that all formulas from $\mathscr{L}_0\cap\Gamma$ will be satisfied in this world. To satisfy also all boxed formulas in Γ we use the inductive hypothesis for every negated box $\neg\Box_J^k\varphi$ in Γ: the application of the rule *($\Box$-Elim)* to that negated box should have at least one underivable denominator (with the corresponding partition δ) and by inductive hypothesis there is a pointed model (M',w_0'), satisfying all formulas in that denominator. Incorporating M' into M and defining distance from w_0 to w_0' on J and all its subcoalitions as the sum of the corresponding values of δ (and as ω for the non-subcoalitions) we ensure that $\Box_J^k\varphi$ is falsified in w_0 while all the non-negated boxes in Γ are satisfied. □

Thus, to check satisfiability of a formula φ in **LGDDA** we can perform an exhaustive search for the closed tableaux in $\textbf{Tab}_{\textbf{LGDDA}}$ for $\{\varphi\}$, for which polynomial space w.r.t. $|\varphi|$ is sufficient.

Theorem 3. *Satisfiability checking in* ***LGDDA*** *is PSPACE-complete.*

Proof. (Sketch) PSPACE-hardness follows from PSPACE-hardenss of logic in [24], which is equivalent to a fragment of **LGDDA**. It can be solved in PSPACE by a proof search in $\textbf{Tab}_{\textbf{LGDDA}}$, starting from $\{\varphi\}$ and trying all possible rule applications. It requires only polynomial space w.r.t. $|\varphi|$ for going through the possible rule applications, and the depth of the proof search is also bounded polynomially. □

6 Conclusion

We have presented a proof-theoretic and complexity analysis of the notion of graded distributed belief, using a formal semantics based on graded belief bases. Following Spohn's ranking theory [32], we plan to study, in future work, a more general variant of the graded belief semantics based on ordinals instead of natural numbers. The notion of graded distributed belief we have defined is based on a counting view (Definition 4). In future work, we intend to investigate a qualitative version by replacing the counting view with a qualitative perspective based on set inclusion. Last but not least, we aim to move from a static to a dynamic setting by extending our framework with a notion of graded belief base change.

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A Tableau calculi and the calculus LK for the classical logic

Semantic tableaux are derivation systems that allow to derive non-satisfiability (and, thus, validity too) in a given logic by decomposing formulas. Semantic tableau *calculus* consists of *rules*, and each rule consist of a set of formulas called *numerator* and a (possibly empty) set of sets of formulas called *denominators*. These rules are usually written as fractions with the numerator above a horizontal line and all denominators separated by the sign '|' — below the horizontal line. Also, usually, a *metarule* is described using metavariables like φ, ψ for arbitrary formulas and X for arbitrary sets of formulas, substituting this variables with specific formulas / sets of formulas we obtain different rule *instances*.

Definition 11. *A tableau in calculus R is a pair (T,L) where T is a tree and a function $L\colon T \to 2^{\mathscr{L}}$ labels each node with a set of formulas in such a way, that for every node $n \in T$ with children $Q \subseteq T$ there exist* *an instance of rule in R such that $L(n)$ is a numerator and $\{L(m) \mid m \in Q\}$ are exactly the denominators of this rule instance. A tableau is closed when $\bot \in L(l)$ for every leaf l in T.*

The intended reading of tableaux rules is that satisfiability of the numerator (i.e. existence of a model satisfying all formulas from it) implies satisfiability of at least one of the denominators. Then the existence of a closed tableau derivation starting from $\{\varphi\}$ implies that φ is not satisfiable, since at such case one of the branches should be labeled with satisfiable sets, but in a closed tableau derivation the labels of the leaves contain $\bot$ and thus can not be satisfiable.

The standard tableau calculus **LK** for classical logic consists of the following four rules, capturing the evaluation of the classical connectives:

$$\frac{\{p,\neg p\}\cup X}{\{\bot\}}\ \textit{(Prop)} \qquad \frac{\{\neg\neg\varphi\}\cup X}{\{\varphi\}\cup X}\ \textit{($\neg$-Elim)}$$

$$\frac{\{\varphi_1\wedge\varphi_2\}\cup X}{\{\varphi_1,\varphi_2\}\cup X}\ \textit{($\wedge$-Elim)} \qquad \frac{\{\neg(\varphi_1\wedge\varphi_2)\}\cup X}{\{\neg\varphi_1\}\cup X \ \mid\ \{\neg\varphi_2\}\cup X}\ \textit{($\vee$-Elim)}$$

B Full proofs

B.1 Proof of Lemma 1

Let $M = (W, \rho, \mathscr{D}, \mathscr{V})$ and let us denote by $Sub\mathscr{F}(\varphi)$ the set of all subformulas of φ. Consider the following equivalence relation on W: $u \equiv_\varphi v$ when $(M,w) \models \psi$ is equivalent to $(M,u) \models \psi$ for all $\psi \in Sub\mathscr{F}(\varphi)$. Now we define a QNGDM $M' = (W', \rho', \mathscr{D}', \mathscr{V}')$ as follows: W' is a set of equivalence classes w.r.t. $\equiv_\varphi$ (we denote equivalence class of $w \in W$ by $\langle w\rangle_\varphi$); $\rho'(J,U,V) \stackrel{def}{=} \min\{\rho(J,u,v) \mid u \in U, v \in V\}$; $\mathscr{D}'(i,\langle w\rangle_\varphi)(\alpha) \stackrel{def}{=} \max^*\{k \mid \triangle_i^k\alpha \in Sub\mathscr{F}(\varphi) \textit{ and } \mathscr{D}(i,w)(\alpha) \geq k\}$; $\mathscr{V}'(p) \stackrel{def}{=} \{\langle w\rangle_\varphi \mid w \in \mathscr{V}(p)\}$. Note that the definition of $\mathscr{D}'$ does not depend on the choice of representative w due to the definition of $\equiv_\varphi$.

We will use the following facts about relating doxastic functions and distances for equivalence classes in M' and their representatives in M: (1) $\rho'(J,\langle w\rangle_\varphi,\langle u\rangle_\varphi) \leq \rho(J,w,u)$ and (2) $\mathscr{D}'(i,\langle w\rangle_\varphi)(\alpha) \leq \mathscr{D}(i,w)(\alpha)$.

Let us show that $(M,w) \models \psi$ iff $(M',\langle w\rangle_\varphi) \models \psi$ for any $\psi \in Sub\mathscr{F}(\varphi)$. We prove it by structural induction on ψ. If ψ is a propositional atom it immediately follows from the definitions of $\mathscr{V}'$ and $\equiv_\varphi$, if top-most connective in ψ is $\neg$ or $\wedge$ then it follows immediately from inductive hypotheses. For $\psi = \triangle_i^l\alpha$, we have $\mathscr{D}'(i,\langle w\rangle_\varphi)(\alpha) \geq l$ iff $\mathscr{D}(i,w)(\alpha) \geq l$ since $\triangle_i^l\alpha \in Sub\mathscr{F}(\varphi)$. For $\psi = \Box_J^k\gamma$, we prove both directions by contraposition. If $(M,w) \not\models \Box_J^k\gamma$ then there is some $u \in W$ such that $\rho(J,w,u) \leq k$ and $(M,u) \not\models \gamma$, then $\rho'(J,\langle w\rangle_\varphi,\langle u\rangle_\varphi) \leq k$ by fact (1) and $(M',\langle u\rangle_\varphi) \not\models \gamma$ by the inductive hypothesis, therefore $(M',\langle w\rangle_\varphi) \not\models \Box_J^k\gamma$. For the opposite direction if $(M',\langle w\rangle_\varphi) \not\models \Box_J^k\gamma$ then there exists U such that $\rho'(J,\langle w\rangle_\varphi,U) \leq k$ and $(M',U) \not\models \gamma$, which means by definition of ρ' that there exist $w_1 \in \langle w\rangle_\varphi, u \in U$ such that $\rho(J,w_1,u) = \rho'(J,\langle w\rangle_\varphi,U) \leq k$ and $(M,u) \not\models \gamma$ by the inductive hypothesis, so $(M,w_1) \not\models \Box_J^k\gamma$ and therefore $(M,w) \not\models \Box_J^k\gamma$ too due to the definition of the equivalence relation.

Now let us check that M' satisfies conditions (QNGDM-DOX) and (QNGDM-ρ-ADD). First, let us show (QNGDM-DOX) for arbitrary $J \in 2^{Agt*}$ and $V,U \in W'$. By definition of ρ' there exist some $v \in V, u \in U$ such that $\rho'(J,V,U) = \rho(J,v,u)$. To get (QNGDM-DOX) for J, V and U in M' from (QNGDM-DOX) for J, v and u in M notice that $Supp(\mathscr{D}'(i,v)) \subseteq Sub\mathscr{F}(\varphi)$ by definition of $\mathscr{D}'$, so for all formulas in $Supp(\mathscr{D}'(i,v))$ we can apply the preservation of evaluation in M' proved in previous paragraph and conclude that the outer sum will iterate through all $\alpha \in Sub\mathscr{F}(\varphi)$ such that $(M,u) \not\models \alpha$, so after moving from M to M' in (QNGDM-DOX) we have a subset of summands in the outer sum and the same summands in the inner sum with $\mathscr{D}(i,v)(\alpha)$ replaced by $\mathscr{D}'(i,V)(\alpha)$ which is not bigger than $\mathscr{D}(i,v)(\alpha)$ by fact (2) above. Thus, the sum on the right may only decrease while the distance on the left is the same, so we have (QNGDM-DOX) for J, V and U. Similarly, we can get (QNGDM-ρ-ADD) for J, V and U in M' from (QNGDM-ρ-ADD) for J, v and u in M: we take the same split on summands δ and it will satisfy the condition for every subcoalition J' since we have $\rho(J',U,V) \leq \rho(J',u,v)$ by fact (1) above.

Thus, M' is a QNGDM and it is also finite: $|W|$ is bounded by number of possible evaluations on subformulas of φ and $Supp(\mathscr{D}(i,w))$ contains only subformulas of φ.

B.2 Proof of Lemma 2

Let $M = (W,\rho,\mathscr{D},\mathscr{V})$ be a QNGDM satisfying φ.
First, we consider $M' = (W',\rho',\mathscr{D}',\mathscr{V}')$, where: $W' = W \times 2^{Agt*}$; $\mathscr{D}'(i,(w,J))(\alpha) = \mathscr{D}(i,w)(\alpha)$; $\mathscr{V}'(p) = \{(w,J) : w \in \mathscr{V}(p)\}$ and

$$\rho'(J',(w,J''),(u,J)) = \begin{cases} \sum_{i\in J'} \delta(i), & \text{if } J' \subseteq J \\ \omega, & \text{otherwise} \end{cases}$$

where δ is taken from the condition (QNGDM-ρ-ADD) for J, w and u. Since M is finite (and we assumed Agt to be a finite set), M' is also finite. Let us show that M' satisfies formula φ, conditions (QNGDM-DOX) and (NGDM-ρ-ADD). By definition of ρ' we have $\rho'(J,(w,J_1),(u,J)) = \rho(J,w,u)$ and $\rho'(J,(w,J_1),(u,J_2)) \geq \rho(J,w,u)$ for any J, J_1, J_2. Due to the latter inequality condition (QNGDM-DOX) for M' follows from (QNGDM-DOX) for M. We also have (NGDM-ρ-ADD) (which is stronger than (QNGDM-ρ-ADD)) for M' by definition of ρ'. What is left to show is that satisfaction of φ in M is preserved in M'. Let us show that $(M,w) \models \varphi$ iff $(M,(w,J'')) \models \varphi$ for any J''. Proof is by structural induction on φ, the only non-trivial case is $\Box$-modality, we prove both directions counterpositively. If $(M,w) \not\models \Box^k_J\psi$ then there exists $u \in W$ such that $(M,u) \not\models \psi$ and $\rho(J,w,u) \leq k$, then by inductive hypothesis $(M',(u,J)) \not\models \psi$ while $\rho(J,(w,J''),(u,J)) = \rho(J,w,u) \leq k$, so $(M',(w,J'')) \not\models \Box^k_J\psi$. If $(M',(w,J'')) \not\models \Box^k_J\psi$ then there exists $(u,J') \in W'$ such that $(M',(u,J') \not\models \psi$ and $\rho'(J,(w,J''),(u,J')) \leq k$, then by inductive hypothesis $(M,u) \not\models \psi$ while $\rho(J,w,u) \leq \rho'(J,(w,J''),(u,J')) \leq k$, so $(M,w) \not\models \Box^k_J\psi$.

Now we will construct an NGDM M'' satisfying formula φ. Let us define the set of relevant atoms as $\Sigma = \mathbb{P}(\varphi) \cup \bigcup_{\substack{i\in Agt\\ w'\in W'\\ \alpha\in Supp(\mathscr{D}'(i,w'))}} \mathbb{P}(\alpha)$ (where $\mathbb{P}(\psi)$ denote the set of all atoms occurring in ψ). Take an aribitrary injective mapping from $\chi\colon W' \to (\mathbb{P}\setminus\Sigma)$ (such mappings exist since $\mathbb{P}$ is infinite while W' and Σ are finite). We take $M'' = (W',\rho',\mathscr{D}'',\mathscr{V}'')$, where:

$$\mathscr{V}''(p) = \begin{cases} W\setminus\{w\}, & \text{if } p = \chi(w) \text{ for some } w \in W' \\ \mathscr{V}'(p), & \text{otherwise} \end{cases}$$

and

$$\mathscr{D}''(i,w)(\alpha) = \begin{cases} \rho'(i,w,u) - \sum\limits_{\substack{\alpha\in\mathscr{L}_0\\ (M',u)\not\models\alpha}} \mathscr{D}'(i,w)(\alpha), & \text{if } \alpha = \chi(u) \text{ for some } u \in W' \\ \mathscr{D}'(i,w)(\alpha), & \text{otherwise} \end{cases}$$

Notice that the evaluation of φ and of any formula from $Supp(\mathscr{D}'(i,w'))$ for $i \in Agt$ and $w' \in W'$ is the same in M' and M'': distances didn't change, the doxastic function and valuation changed only on atoms that do not occur in such formulas. Let us now show that M'' satisfies (NGDM-DOX). Since M' satisfies (NGDM-ρ-ADD) it is sufficient to check it for one-agent coaltitions, for any other coalition J we have sum over all $i \in J$. Take arbitrary $i \in Agt$, $w', u' \in W'$, we need to show (*) $\rho'(\{i\}, w', u') = \sum_{\substack{\alpha \in \mathscr{L}_0 \\ (M'', u') \not\models \alpha}} \mathscr{D}''(i, w')(\alpha)$. Those summands on the right that has $\alpha \in Supp(\mathscr{D}'(i, w'))$ did not change: for them $\mathscr{D}''(i, w') = \mathscr{D}'(i, w')$ and $(M'', u') \not\models \alpha$ is equivalent to $(M', u') \not\models \alpha$, so this part of the sum equals to $\sum_{\substack{\alpha \in \mathscr{L}_0 \\ (M', u') \not\models \alpha}} \mathscr{D}'(i, w')(\alpha)$. Summands for $\alpha \notin Supp(\mathscr{D}'(i, w'))$ can be non-zero only if $\alpha = \chi(v')$ for some $v' \in W'$ (for others $\mathscr{D}''(i, w')(\alpha) = \mathscr{D}'(i, w')(\alpha) = 0$). At the same time, $(M'', u') \not\models \chi(v')$ is equivalent to $v' = u'$ by the definition of $\mathscr{V}''$, therefore then only additional potentially non-zero summand will be $\mathscr{D}''(i, w')(\chi(u')) = \rho'(\{i\}, w', u') - \sum_{\substack{\alpha \in \mathscr{L}_0 \\ (M', u) \not\models \alpha}} \mathscr{D}'(i, w)(\alpha)$. Thus, the sum on the right of (*) equals $\rho'(\{i\}, w', u')$, as required.

B.3 Proof of Lemma 3

Suppose there is an NGDM $M^{NGDM} = (W, \mathscr{D}, \rho, \mathscr{V})$ such that $(M^{NGDM}, w_0) \models \varphi$ for some $w_0 \in W$. Consider the following mapping from W to $\mathbf{S}$: $g(w) = (\mathscr{B}_1, \ldots, \mathscr{B}_n, V)$ where $\mathscr{B}_i(\alpha) = \mathscr{D}(i, w)(\alpha)$ for all $i \in Agt$ and $\alpha \in \mathscr{L}_0$ and $V = \{p \in \mathbb{P} : w \in \mathscr{V}(p)\}$. Then we can take $(g(w_0), g(W))$ as a satisfying model, where $g(W) = \{g(w) \mid w \in W\}$. The fact that $(g(w), g(W)) \models \psi$ iff $(M^{NGDM}, w) \models \psi$ can be established by simple structural induction, for the cases of boxes $\rho(J, w, u) \leq k$ is equivalent to $g(w)\mathscr{R}^k_J g(u)$ due to the condition (NGDM-DOX).

B.4 Proof of Lemma 4

Suppose there is an MAGBM $M^{MAGBM} = (S, U)$ such that $(S, U) \models \varphi$. We define a satisfying QNGDM $M^{QNGDM} = (W, \mathscr{D}, \rho, \mathscr{V})$ as follows: $W \overset{def}{=} \{S\} \cup U$; $\mathscr{D}(i, (\mathscr{B}_1, \ldots, \mathscr{B}_n, V))(\alpha) \overset{def}{=} \mathscr{B}_i(\alpha)$; $\mathscr{V}(p) \overset{def}{=} \{(\mathscr{B}_1, \ldots, \mathscr{B}_n, V) \in W : p \in V\}$; and

$$\rho(J, S, S') \overset{def}{=} \begin{cases} \sum_{\substack{\alpha \in \mathscr{L}_0 \\ (M^{QNGDM}, S') \not\models \alpha}} \sum_{i \in J} \mathscr{D}(i, S)(\alpha) & \text{if } S' \in U \\ \omega & \text{otherwise} \end{cases}$$

M^{QNGDM} satisfies (NGDM-DOX) (and therefore both (QNGDM-DOX) and (QNGDM-ρ-ADD)) when $S' \in U$ and has infinite distance (for which both condition do not apply by definition) otherwise. Also, $(M^{QNGDM}, S') \models \psi$ iff $(S', U) \models \psi$ for every $\psi \in \mathscr{L}$, which can be proved by structural induction. The only interesting case is that of the box, in which case doxastic alternatives correspond precisely to the states not further than the given distance (and only those belonging to the context since the given distance is always finite).

B.5 Proof of Theorem 1

Soundness. Soundness of all axioms and rules apart from ($\mathbf{Int}_{\triangle_i^{k'}, \Box_J^k}$) and ($\mathbf{Int}_{\Box_{J'}^{k'}, \Box_J^k}$) is straightforward.

Soundness of ($\mathbf{Int}_{\triangle_i^{k'}, \Box_J^k}$) follows from condition (QNGDM-DOX). Suppose $M, w \models \bigwedge_{\triangle_i^{k'} \alpha \in \Omega} \triangle_i^{k'} \alpha$, then $\mathscr{D}(i, w)(\alpha) \geq k'$ for all $\triangle_i^{k'} \alpha \in \Omega$. Now consider any $u \in W$ such that $\rho(J, w, u) \leq k$. Let us denote $\Omega' = \{\triangle_i^{k'}\alpha \in \Omega \mid M,u \not\models \alpha\}$. Then clearly $M,u \models \bigwedge_{\triangle_i^{k'}\alpha\in\Omega\setminus\Omega'}\alpha$ and this conjunction is one of the disjuncts in $(\mathbf{Int}_{\triangle_i^{k'},\Box_J^k})$ since (due to (QNGDM-DOX) and the side condition of the axiom) $\sum\{k' \mid \triangle_i^{k'}\alpha \in \Omega'\} \le \sum_{\substack{\alpha\in\mathscr{L}_0\\(M,u)\not\models\alpha}}\sum_{i\in J}\mathscr{D}(i,w)(\alpha) \le \rho(J,u,w) \le k$.

Soundness of $(\mathbf{Int}_{\Box_{J'}^{k'},\Box_J^k})$ follows from condition (QNGDM-ρ-ADD). Suppose $M,w \models \bigwedge_{\Box_{J'}^{k'}\psi\in\Psi}\Box_{J'}^{k'}\psi$. Consider any $u \in W$ such that $\rho(J,w,u) \le k$. Take $\delta \in \mathscr{P}art(\rho(J,w,u),J)$ from (QNGDM-ρ-ADD), there exists some $\delta' \in \mathscr{P}art(k,J)$ such that $\delta'(i) \ge \delta(i)$ for every $i \in J$ and the disjunct in $(\mathbf{Int}_{\Box_{J'}^{k'},\Box_J^k})$ corresponding to δ' is satisfied in u if the antecedent of $(\mathbf{Int}_{\Box_{J'}^{k'},\Box_J^k})$ is satisfied in w: if $\Box_{J'}^{k'}\psi \in \Psi$ and $\sum_{i\in J'}\delta'(i) \le k'$ then (by (QNGDM-ρ-ADD)) $\rho(J',w,u) \le \sum_{i\in J'}\delta(i) \le \sum_{i\in J'}\delta'(i) \le k'$ so $M,u \models \psi$.

Completeness. We prove completeness following the standard canonical model construction approach for modal logics [4]. In particular we consider **LGDDA**-maxiconsistent sets of formulas: $\mathscr{E} = \{\Phi \subseteq \mathscr{L} \mid \Phi \not\vdash_{\mathbf{LGDDA}} \bot \textit{ and for any } \varphi \in \mathscr{L} \textit{ either } \varphi \in \Phi \textit{ or } \neg\varphi \in \Phi\}$. As usual, for such sets we have $\varphi \in \Phi \Leftrightarrow \Phi \vdash_{\mathbf{LGDDA}} \varphi$ (obvious from left to right, and inverse direction is given be the fact that derivability of both φ and $\neg\varphi$ would imply derivability of $\bot$) and the fact that any set consistent w.r.t. **LGDDA** can be extended to a set from $\mathscr{E}$ (standard Lindenbaum's lemma [4]).

We define a canonical QNGDM $M^C = (W^C,\mathscr{D}^C,\rho^C,\mathscr{V}^C)$ as follows:

- $W^C = \mathscr{E}$
- $\mathscr{D}^C(i,\Phi)(\alpha) = \max^*\{k\colon \triangle_i^k\alpha \in \Phi\}$
- $\rho^C(J,\Phi,\Phi') = \min^*\{k\colon \Box_J^k\varphi \in \Phi \Rightarrow \varphi \in \Phi'\}$
- $\mathscr{V}^C(p) = \{\Phi \in W^C\colon p \in \Phi\}$

We will show that this is indeed a QNGDM (i.e. that it satisfies conditions (QNGDM-DOX) and (QNGDM-ρ-ADD)) later, but first we prove the truth lemma for the canonical model, i.e. $M^C,\Phi \models \varphi$ iff $\varphi \in \Phi$. We prove the truth lemma by structural induction on formula φ. The case of propositional atom follows from the definiton of $\mathscr{V}^C$, the case of negation — from the inductive hypothesis and definition of $\mathscr{E}$, the case of conjunction — from the inductive hypotheses and the fact that derivability of $\psi_1 \wedge \psi_2$ in classical logic is equivalent to derivability of both ψ_1 and ψ_2. If $\triangle_i^k\alpha \in \Phi$ then $\mathscr{D}^C(i,\Phi)(\alpha) \ge k$ so $M^C,\Phi \models \triangle_i^k\psi$. Conversely, if $M^C,\Phi \models \triangle_i^k\psi$ then $\mathscr{D}^C(i,\Phi)(\alpha) \ge k$ so there is some $\triangle_i^{k'}\alpha \in \Phi$ such that $k' \ge k$, then $\triangle_i^k\alpha \in \Phi$ using the axiom $(\mathbf{Mon}_{\triangle_i^l})$. If $\Box_J^k\psi \in \Phi$ and $\rho^C(J,\Phi,\Phi') \le k$ for some Φ' then $\Box_J^{\rho^C(J,\Phi,\Phi')}\psi \in \Phi$ using validity $(\mathbf{Mon}^k_{\Box_J^k})$ (which is an instance of $(\mathbf{Int}_{\Box_{J'}^{k'},\Box_J^k})$), which implies by definition of ρ^C that $M^C,\Phi' \models \psi$, thus $M^C,\Phi \models \Box_J^k\psi$. Conversely, if $\Box_J^k\psi \notin \Phi$ then $\Phi^{\Box_J^k\downarrow} \not\vdash_{\mathbf{LGDDA}} \psi$ for $\Phi^{\Box_J^k\downarrow} = \{\xi : \Box_J^k\xi \in \Phi\}$, since otherwise there would be a finite $\Xi \subseteq \Phi^{\Box_J^k\downarrow}$ such that $(\bigwedge_{\xi\in\Xi}\xi) \to \psi$, which together with rule $(\mathbf{Nec}_{\Box_J^k})$, axiom $(\mathbf{K}_{\Box_J^k})$ and modus ponens implies $\Phi \vdash_{\mathbf{LGDDA}} \Box_J^k\psi$ and therefore $\Box_J^k\psi \in \Phi$. Therefore the set $\Phi^{\Box_J^k\downarrow} \cup \{\neg\psi\}$ is consistent w.r.t. **LGDDA** and can therefore be extended to a complete consistent set Φ' by Lindenbaum's lemma, and $M^C,\Phi' \not\models \psi$ (by inductive hypotheis) and $\rho^C(J,\Phi,\Phi') \le k$ (by the definitions of ρ^C and Φ'), so $M^C,\Phi \not\models \Box_J^k\psi$.

Now we can show that M^C satisfies (QNGDM-DOX) and (QNGDM-ρ-ADD) and is therefore a QNGDM. First, we will prove (QNGDM-DOX) by contradiction. Suppose towards contradiction that $\rho^C(J,\Phi,\Phi') < \sum_{\substack{\alpha\in\mathscr{L}_0\\(M,u)\not\models\alpha}}\sum_{i\in J}\mathscr{D}^C(i,w)(\alpha)$. Then by the definition of $\mathscr{D}^C$ we can select finite $\Omega \subseteq \Phi$ such that all formulas in Ω have the form $\triangle_i^k\alpha$ for some $i \in J$ and some $\alpha \in \mathscr{L}_0$ such that $(M^C,\Phi') \not\models \alpha$, and such that there is at most one formula in Ω for every α and i and such that $\sum\{k\colon \triangle_i^k\alpha \in \Omega\} > \rho^C(J,\Phi,\Phi')$. We can take instance of $(\mathbf{Int}_{\triangle_i^{k'},\Box_J^k})$ for coalition J, grade $\rho^C(J,\Phi,\Phi')$ and our selected Ω

and apply modus ponens to it (since $\Omega \subseteq \Phi$) to get $\Box_J^{\rho^C(J,\Phi,\Phi')} D \in \Phi$ where $D = \bigvee_{\substack{\Omega' \subseteq \Omega \\ \sum\{l \mid \triangle_i^l \alpha \in \Omega'\} \leq \rho^C(J,\Phi,\Phi')}} \bigwedge_{\triangle_i^l \alpha \in \Omega \setminus \Omega'} \alpha$.
By definition of $\rho^C(J,\Phi,\Phi')$ we then have $D \in \Phi'$, which by the truth lemma implies $M^C, \Phi' \models D$, which means that there exists $\Omega' \subseteq \Omega$ such that $\sum\{k' \mid \triangle_i^{k'} \alpha \in \Omega'\} \leq \rho^C(J,\Phi,\Phi')$ and such that $M^C, \Phi' \models \alpha$ for every $\triangle_i^{k'} \alpha \in \Omega \setminus \Omega'$. But since $\sum\{k' \mid \triangle_i^{k'} \alpha \in \Omega\} > \rho^C(J,\Phi,\Phi')$ set $\Omega \setminus \Omega'$ is not empty, but at the same time if $\triangle_i^{k'} \alpha \in (\Omega \setminus \Omega') \subseteq \Omega$ then $M^C, \Phi' \not\models \alpha$ by the choice of Ω, which leads to desired contradiction.

Now we prove (QNGDM-ρ-ADD) also by contradiction. Suppose for some $J \in 2^{Agt*}$ and $\Phi, \Phi' \in \mathscr{E}$ condition (QNGDM-ρ-ADD) fails, i.e. for any partition $\delta \in \mathscr{P}art(\rho^C(J,\Phi,\Phi'),J)$ there exists a subcoalition $J_\delta \subset J$ such that $d_\delta = \sum_{i \in J_\delta} \delta(i) < \rho^C(J_\delta,\Phi,\Phi')$. By definition of ρ^C for any such subcoalition there should exists $\psi_\delta \in \mathscr{L}$ such that $\Box_{J_\delta}^{d_\delta} \psi_\delta \in \Phi$ but $\psi_\delta \notin \Phi'$. Take $\Omega = \{\Box_{J_\delta}^{d_\delta} \psi_\delta \mid \delta \in \mathscr{P}art(\rho^C(J,\Phi,\Phi'),J)\}$, such Ω is finite (since $\rho^C(J,\Phi,\Phi')$ is finite in (QNGDM-ρ-ADD)). We can take the instance of ($\mathbf{Int}_{\triangle_i^{k'}, \Box_J^k}$) for coalition J, grade $\rho^C(J,\Phi,\Phi')$ and our selected Ω and apply modus ponens to it (since $\Omega \subseteq \Phi$) to get $\Box_J^{\rho^C(J,\Phi,\Phi')} D \in \Phi$ where $D = \bigvee_{\delta \in \mathscr{P}art(k,J)} \bigwedge_{\substack{\Box_{J'}^{k'} \psi \in \Psi \\ \sum_{i \in J'} \delta(i) \leq k'}} \psi$. By definition of $\rho^C(J,\Phi,\Phi')$ we then have $D \in \Phi'$, which by the truth lemma implies $M^C, \Phi' \models D$, which means that there exists partition $\delta \in \mathscr{P}art(\rho^C(J,\Phi,\Phi'),J)$ such that $M^C, \Phi' \models \psi$ for every $\Box_{J'}^{k'} \psi \in \Omega$ such that $\sum_{i \in J'} \delta(i) \leq k'$. But $\Box_{J_\delta}^{d_\delta} \psi_\delta \in \Omega$ and $\sum_{i \in J_\delta} \delta(i) = d_\delta$ by definition of d_δ, while $M^C, \Phi' \not\models \psi_\delta$ by the choice of ψ_δ, which leads to desired contradiction.

The truth lemma implies completeness: if $\Gamma \not\vdash_{\mathbf{LGDDA}} \varphi$ for some $\Gamma \subseteq \mathscr{L}$, then $\Gamma \cup \{\neg\varphi\}$ is consistent w.r.t. **LGDDA** and can be extended to a set $\Phi \in \mathscr{E}$, and by truth lemma $M^C, \Phi \not\models \varphi$ and $M^C, \Phi \not\models \psi$ for any $\psi \in \Gamma$, so $\Gamma \not\models_{QNGDM} \varphi$.

B.6 Proof of Theorem 2

Soundness: Let us show that existence of the closed tableau for Γ implies that all formulas from Γ can not be satisfied simultaneously. The proof proceeds by structural induction on the closed tableau. For the base case (leaf), Γ contains $\bot$ which can not be satisfied. For the inductive case we check that for every rule satisfiability of the numerator implies satisfiability of at least one denominator (which is false by inductive hypothesis). For rules apart from *($\Box$-Elim)* this is straightforward. For rule *($\Box$-Elim)* suppose there exists a QNGDM M with the world w that satisfies all formulas in $\{\neg\Box_J^k \varphi\} \cup X$. Then there exists a world u in M such that $\rho(J,w,u) \leq k$ and $M, u \not\models \varphi$. By (QNGDM-ρ-ADD) there exists $\delta \in \mathscr{P}art(\rho(J,w,u),J)$ such that $\sum_{i \in J'} \delta(i) \geq \rho(J',w,u)$ for every $J' \subset J$. Consider the subset of X defined as $Y = \{\Box_{J'}^{k'} \psi \colon \sum_{i \in J'} \delta(i) \leq k'\} \cup \{\triangle_i^k \alpha \colon M, u \models \alpha\}$. Y is one of subests $\{Y_1, \ldots, Y_N\}$ appearing in the rule (due to (QNGDM-DOX)), and there is a denominator corresponding to it. All formulas in this denominator are satisfied in the world u in M. $\neg\varphi$ is satisfied by the choice of u, $Y^{\Box\downarrow}$ and $Y^{\Delta_J\downarrow}$ are satisfied by choice of Y and δ (ensuring $\rho(J',w,u) \leq k'$ for each $\Box_{J'}^{k'} \psi \in Y$).
Completeness: We will show that any non-derivable Γ (i.e. Γ for which there does not exist a closed tableaux) can be satisfied in some pointed QNGDM satisfying (NGDM-ρ-ADD). We construct such a QNGDM inductively, via induction on the number of connectives and modalities in Γ. We will rely on the fact that since Γ is non-derivable, then for any possible instance of a rule with Γ in the numerator, at least one of the denominators has to be non-derivable too (otherwise we would obtain a closed tableau).

First, suppose that Γ contains a formula of the form $\neg\neg\varphi$, then we can apply the rule *($\neg$-Elim)* and infer that the denominator $\Gamma \setminus \{\neg\neg\varphi\} \cup \{\varphi\}$ is also non-derivable. By inductive hypothesis (denominator contains fewer connectivess) there exists a pointed QNGDM (M,w) such that $M, w \models \varphi$ and $M, w \models \psi$

for all $\psi \in \Gamma \setminus \{\neg\neg\varphi\}$, this model thus satisfies all formulas in Γ. We can handle similarly the cases when some formula of the form $\varphi_1 \wedge \varphi_2$ or of the form $\neg(\varphi_1 \wedge \varphi_2)$ appears in Γ: by applying the rule *($\wedge$-Elim)* or *($\vee$-Elim)* respectively, and then applying the inductive hypothesis to the non-derivable denominator, we obtain a model satisfying Γ. Now, the remaining case is when Γ contains only formulas of the forms $p \in \mathbb{P}$, $\neg p$, $\triangle_i^k \alpha$, $\neg\triangle_i^k \alpha$, $\Box_J^k \varphi$ or $\neg\Box_J^k \varphi$. We then construct a model satisfying Γ by putting together inductively obtained models for each negative box in Γ and adding one additional world where exactly the atoms and triangles that belong to Γ will be satisfied. Specifically, for each $\neg\Box_J^k \varphi \in \Gamma$ consider application of the rule *($\Box$-Elim)* to Γ and this negative box. There should exist a set $Y_{\Box_J^k \varphi} \subseteq \Gamma \setminus \{\neg\Box_J^k \varphi\}$ and a partition $\delta_{\Box_J^k \varphi} \in \mathscr{P}art(k,J)$ such that the corresponding denominator $\{\neg\varphi, Y_{\Box_J^k \varphi}^{\Box\downarrow}, Y_{\Box_J^k \varphi}^{\Delta_J\downarrow}\}$ is non-derivable. By inductive hypothesis there exists a pointed QNGDM $(M_{\Box_J^k \varphi}, w_{\Box_J^k \varphi})$ satisfying (NGDM-ρ-ADD) that satisfies all formulas in this denominator. We construct a pointed QNGDM (M, w_0) satisfying condition (NGDM-ρ-ADD) and satisfying all formulas in Γ as follows:

- The set of worlds in M is disjoint union of worlds in $M_{\Box_J^k \varphi}$ for all $\neg\Box_J^k \varphi \in \Gamma$ (we will refer to them as *submodels*), plus one additional distinctive world w_0.
- The doxastic function $\mathscr{D}$ are not changed in any world of any submodel $M_{\Box_J^k \varphi}$, and for w_0 it is defined as $\mathscr{D}(i, w_0)(\alpha) = \max^*\{k' : \triangle_i^{k'} \alpha \in \Gamma\}$.
- The distance function is defined as follows:

$$\rho(J', w, u) = \begin{cases} \rho_{M_{\Box_J^k \varphi}}(J', w, u), & \textit{if both } w \textit{ and } u \textit{ belong to a submodel } M_{\Box_J^k \varphi} \\ \sum_{i \in J'} \delta_{M_{\Box_J^k \varphi}}(i), & \textit{if } w = w_0 \textit{ and } u = w_{M_{\Box_J^k \varphi}} \textit{ and } J' \subseteq J \\ \omega, & \textit{otherwise} \end{cases}$$

- The valuation is preserved on submodels and defined on w_0 in such a way, that it satisfies exactly the atoms in $\Gamma \cap \mathbb{P}$.

Notice that for every $w \in M_{\Box_J^k \varphi}$, $M_{\Box_J^k \varphi}, w \models \varphi$ iff $M, w \models w$ (since all worlds outside $M_{\Box_J^k \varphi}$ in M are on the infinite distance from w, and therefore do not affect evaluation of any formula). All formulas in Γ are satisfied in (M, w_0): for $p \in \Gamma \cap \mathbb{P}$ we have $M, w_0 \models p$ by definition of valuation and for $\neg p \in \Gamma$ we have $p \notin \Gamma$ (otherwise we can obtain a closed tableau by applying rule *(Prop)*) so $M, w_0 \not\models p$; for $\triangle_i^k \alpha \in \Gamma$ we have $M, w_0 \models \triangle_i^k \alpha$ by definition of the doxastic function and for $\neg\triangle_i^{k'} \alpha \in \Gamma$ there is no $\neg\triangle_i^k \alpha \in \Gamma$ with $k \geq k'$ (otherwise we can obtain a closed tableau by applying rule *($\triangle$-Mon)*) so $M, w_0 \not\models \triangle_i^{k'} \alpha$ by definition of the doxastic function; for $\neg\Box_J^k \varphi \in \Gamma$ we have $M, w_0 \not\models \Box_J^k \varphi$ since $M, w_{\Box_J^k \varphi} \not\models \varphi$ and $\rho(J, w_0, w_{\Box_J^k \varphi}) = \sum_{i \in J} \delta_{\Box_J^k \varphi}(i) = k$; finally, for any $\Box_{J'}^{k'} \psi \in \Gamma$ if $\rho(J', w_0, w') \leq k'$ then $w' = w_{\Box_J^k \varphi}$ for some $J' \subseteq J$ and $\rho(J', w_0, w_{\Box_J^k \varphi}) = \sum_{i \in J'} \delta_{M_{\Box_J^k \varphi}}(i)$, which implies that ψ is present in $Y_{\Box_J^k \varphi}^{\Box\downarrow}$ and therefore ψ is satisfied in $w_{\Box_J^k \varphi}$, thus $M, w_0 \models \Box_{J'}^{k'} \psi \in \Gamma$.

We also need to check that M satisfies conditions (QNGDM-DOX) and (NGDM-ρ-ADD). It satisfies (NGDM-ρ-ADD) by the definition of the distance function and due to the fact (given by inductive hypothesis) that (NGDM-ρ-ADD) is satisfied in every submodel $M_{\Box_J^k \varphi}$. It satisfies (QNGDM-DOX) for two worlds inside one submodel by inductive hypothesis (since both the doxastic function and the distance function are preserved in M), for distance $\rho(J', w_0, w_{M_{\Box_J^k \varphi}})$ for $J' \subseteq J$ since all formulas in $Y_{\Box_J^k \varphi}^{\Delta_J\downarrow}$ are satisfied in $w_{M_{\Box_J^k \varphi}}$ and therefore $\sum_{\substack{\alpha \in \mathscr{L}_0 \\ (M, w_{\Box_J^k \varphi}) \not\models \alpha}} \sum_{i \in J'} \mathscr{D}(i, w_0)(\alpha) \leq \sum_{i \in J'} \mathscr{D}_{agg}(i, \Gamma \setminus Y_{\Box_J^k \varphi}) \leq \sum_{i \in J'} \delta_{\Box_J^k \varphi}(i) = \rho(J', w_0, w_{\Box_J^k \varphi})$, and for all other cases since the distance is infinite.

B.7 Proof of Theorem 3

PSPACE-hardness. To prove that satisfiability checking in **LGDDA** is PSPACE-hard, we reduce satisfiability in logic **LDA** (proved to be PSPACE-complete in [26]) to it. In fact, **LDA** is equivalent to a fragment of **LGDDA** with zero degrees and singleton coaltitions, i.e. with the translation $t : \mathscr{L}_{\mathbf{LDA}} \to \mathscr{L}_{\mathbf{LGDDA}}$ defined as

$$\begin{array}{lcl} t(p) & = & p \\ t(\neg\varphi) & = & \neg t(\varphi) \\ t(\varphi_1 \wedge \varphi_2) & = & t(\varphi_1) \wedge t(\varphi_2) \\ t(\triangle_i \alpha) & = & \triangle_i^1 t(\alpha) \\ t(\Box_i \varphi) & = & \Box_{\{i\}}^0 t(\varphi) \end{array}$$

the satisfability of φ in **LDA** is equivalent to satisfability of $t(\varphi)$ in **LGDDA**: we can easily map MAGBM-models with multiset belief bases into their **LDA**-versions with set belief bases by changing all cardinalities to 1, and have an identity mapping back, and both of these mappings preserve satisfaction of formulas from this fragment since there are no non-trivial degrees.

PSPACE-membership. According to Th. 2 checking (non-)satisfiability of φ is equivalent to checking the existence of a closed tableau starting from the set $\{\varphi\}$. We can check the existence of a closed tableau starting from an arbitrary set Γ via exhaustive search with the recursive algorithm in Fig. 1.

This algorithm attempts all possible applications of the rules from $\mathbf{Tab_{LGDDA}}$: rule *(Prop)* in **for**-cycle on line 2, rule *(¬-Elim)* in **for**-cycle on line 5, rule *(∧-Elim)* in **for**-cycle on line 8, rule *(∨-Elim)* in **for**-cycle on line 11, rule *(△-Mon)* in **for**-cycle on line 14, and rule *(□-Elim)* in **for**-cycle on line 17; and recursively searches for closed tableu derivations for all denominators of the attempted rule instance. In case there are closed tableau derivations for all denominators of a given rule instance (and therefore a closed tableaux for Γ starting with this rule instance) **True** is returned, and if after checking all possible first rule instances no tableau derivation is found **False** is returned on line 30. When trying the applications of the rule *(□-Elim)* in the **for**-cycle on line 17, for every subset $Y \subseteq \Gamma \setminus \{\neg\Box_J^k \varphi\}$ it is first checked whether the corresponding partition δ from the rule definition exists (by iterating through all possible partitions) and only in such case the recursive search for denominators is performed.

Notice that inside the function `Has_Closed_Tableau`(Γ) the polynomial amount of space w.r.t. $|\Gamma|$ is used: subset Y can be characterized with $|\Gamma|$ bits and partition of k on J agents can be characterized with $|J| \leq |Agt|$ numbers not exceeding k, and k appear in the input. Also, with the rules from $\mathbf{Tab_{LGDDA}}$ the size of Γ (measured in connectives and modalities appearing in it) decreases with each recursive call, so the depth of recursion is linearly bounded w.r.t. the size of the input.

```
function HAS_CLOSED_TABLEAU(Γ)
    for p ∈ (ℙ ∩ Γ) do
        if ¬p ∈ Γ then
            return True
    for ¬¬φ ∈ Γ do
        if HAS_CLOSED_TABLEAU(Γ \ {¬¬φ} ∪ {φ}) then
            return True
    for φ1 ∧ φ2 ∈ Γ do
        if HAS_CLOSED_TABLEAU(Γ \ {φ1 ∧ φ2} ∪ {φ1, φ2}) then
            return True
    for ¬(φ1 ∧ φ2) ∈ Γ do
        if HAS_CLOSED_TABLEAU(Γ \ {¬(φ1 ∧ φ2)} ∪ {¬φ1})
        and HAS_CLOSED_TABLEAU(Γ \ {¬(φ1 ∧ φ2)} ∪ {¬φ2}) then
            return True
    for △_i^k α ∈ Γ do
        if ¬△_i^{k'} α ∈ Γ and k ≥ k' then
            return True
    for ¬□_J^k φ ∈ Γ do
        All_Denominators_Closed ← True
        for Y ⊆ Γ \ {¬□_J^k φ} do
            Exists_Partition ← False
            for δ ∈ 𝒫art(k, J) do
                if ∀□_{J'}^{k'} ψ ∈ Y holds ∑_{i∈J'} δ(i) ≤ k'
                and ∀i ∈ J holds 𝒟_agg(i, X \ Y) ≤ δ(i) then
                    Exists_Partition ← True
            if Exists_Partition then
                Y↓ ← {ψ | □_{J'}^{k'} ψ ∈ Y} ∪ {α | △_i^{k'} α ∈ Y, i ∈ J}
                if not HAS_CLOSED_TABLEAU({¬φ} ∪ Y↓) then
                    All_Denominators_Closed ← False
        if All_Denominators_Closed then
            return True
    return False
```

Figure 1: Proof search algorithm for $\mathbf{Tab_{LGDDA}}$.